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3 (Sem-2/CBCS) MAT HC 1

2022

**MATHEMATICS**

(Honours)

Paper : MAT-HC-2016

**(Real Analysis)**

Full Marks : 80

Time : Three hours

**The figures in the margin indicate full marks for the questions.**

1. Answer **any ten** questions :  $1 \times 10 = 10$

(a) Find the infimum of the set

$$\left\{ 1 - \frac{(-1)^n}{n} : n \in \mathbb{N} \right\}$$

(b) If  $A$  and  $B$  are two bounded subsets of  $\mathbb{R}$ , then which one of the following is true?

(i)  $\sup(A \cup B) = \sup\{\sup A, \sup B\}$

(ii)  $\sup(A \cup B) = \sup A + \sup B$

Contd.

(iii)  $\sup(A \cup B) = \sup A \cdot \sup B$

(iv)  $\sup(A \cup B) = \sup A \cup \sup B$

(c) There does not exist a rational number  $x$  such that  $x^2 = 2$ . (Write True or False)

(d) The set  $\mathbb{Q}$  of rational numbers is uncountable. (Write True or False)

(e) If  $I_n = \left(0, \frac{1}{n}\right)$  for  $n \in \mathbb{N}$ , then  $\bigcap_{n=1}^{\infty} I_n = ?$

(f) The convergence of  $\{|x_n|\}$  imply the convergence of  $\{x_n\}$ .  
(Write True or False)

(g) What are the limit points of the sequence  $\{x_n\}$ , where  $x_n = 2 + (-1)^n$ ,  $n \in \mathbb{N}$  ?

(h) If  $\{x_n\}$  is an unbounded sequence, then there exists a properly divergent subsequence. (Write True or False)

(i) A convergent sequence of real numbers is a Cauchy sequence.  
(Write True or False)

(j) If  $0 < a < 1$  then  $\lim_{n \rightarrow \infty} a^n = ?$

(k) The positive term series  $\sum \frac{1}{n^p}$  is convergent if and only if

(i)  $p > 0$

(ii)  $p > 1$

(iii)  $0 < p < 1$

(iv)  $p \leq 1$

*(Write correct one)*

(l) Define conditionally convergent of a series.

(m) If  $\{x_n\}$  is a convergent monotone

sequence and the series  $\sum_{n=1}^{\infty} y_n$  is

convergent, then the series  $\sum_{n=1}^{\infty} x_n y_n$  is

also convergent.

*(Write True or False)*

(n) Consider the series  $\sum_{n=1}^{\infty} \frac{1}{n^m \left(1 + \frac{1}{n^p}\right)}$

where  $m$  and  $p$  are real numbers under which of the following conditions does the above series convergent ?

(i)  $m > 1$

(ii)  $0 < m < 1$  and  $p > 1$

(iii)  $0 \leq m \leq 1$  and  $0 \leq p \leq 1$

(iv)  $m = 1$  and  $p > 1$

(o) Let  $\{x_n\}$  and  $\{y_n\}$  be sequences of real

numbers defined by  $x_1 = 1$ ,  $y_1 = \frac{1}{2}$ ,

$$x_{n+1} = \frac{x_n + y_n}{2} \text{ and } y_{n+1} = \sqrt{x_n y_n} \quad \forall n \in \mathbb{N}$$

then which one of the following is true ?

(i)  $\{x_n\}$  is convergent, but  $\{y_n\}$  is not convergent

(ii)  $\{x_n\}$  is not convergent, but  $\{y_n\}$  is convergent

(iii) Both  $\{x_n\}$  and  $\{y_n\}$  are convergent  
and  $\lim_{n \rightarrow \infty} x_n > \lim_{n \rightarrow \infty} y_n$

(iv) Both  $\{x_n\}$  and  $\{y_n\}$  are convergent  
and  $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n$

2. Answer **any five** parts :  $2 \times 5 = 10$

(a) If  $a$  and  $b$  are real numbers and if  $a < b$ ,  
then show that  $a < \frac{1}{2}(a + b) < b$ .

(b) Show that the sequence  $\left\{ \frac{2n - 7}{3n + 2} \right\}$  is  
bounded.

(c) If  $\{x_n\}$  converges in  $\mathbb{R}$ , then show that  
 $\lim_{n \rightarrow \infty} x_n = 0$

(d) Show that the series  $1 + 2 + 3 + \dots$ , is not  
convergent.

(e) Test the convergence of the series :

$$1 - \frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \dots$$

(f) State Cauchy's integral test of convergence.

(g) State the completeness property of  $\mathbb{R}$  and find the  $\sup \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$ .

(h) Does the Nested Interval theorem hold for open intervals? Justify with a counter example.

3. Answer **any four** parts : 5×4=20

(a) If  $x$  and  $y$  are real numbers with  $x < y$ , then prove that there exists a rational number  $r$  such that  $x < r < y$ .

(b) Show that a convergent sequence of real numbers is bounded.

(c) Prove that  $\lim_{n \rightarrow \infty} \left( n^{\frac{1}{n}} \right) = 1$ .

(d)  $\{x_n\}$  be a sequence of real numbers that converges to  $x$  and suppose that  $x_n \geq 0$ . Show that the sequence  $\{\sqrt{x_n}\}$  of positive square roots converges and  $\lim_{n \rightarrow \infty} \sqrt{x_n} = \sqrt{x}$ .

(e) Show that every absolutely convergent series is convergent. Is the converse true? Justify.  $4+1=5$

(f) Using comparison test, show that the series  $\sum (\sqrt{n^4 + 1} - \sqrt{n^4 - 1})$  is convergent.

(g) State Cauchy's root test. Using it, test the convergence of the series

$$\left(\frac{2^2}{1^2} - \frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3} - \frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4} - \frac{4}{3}\right)^{-3} + \dots$$

$$1+4=5$$

(h) Show that the sequence defined by the recursion formula

$$u_{n+1} = \sqrt{3u_n}, \quad u_1 = 1$$

is monotonically increasing and bounded. Is the sequence convergent?

$$2+2+1=5$$

4. Answer **any four** parts : 10×4=40

(a) Show that the sequence  $\left\{ \left(1 + \frac{1}{n}\right)^n \right\}$  is

convergent and  $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$  which lies between 2 and 3.

(b) (i) Let  $\{x_n\}$ ,  $\{y_n\}$  and  $\{z_n\}$  are sequences of real numbers such that  $x_n \leq y_n \leq z_n$  for all  $n \in \mathbb{N}$  and

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} z_n.$$

Show that  $\{y_n\}$  is convergent and

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n = \lim_{n \rightarrow \infty} z_n \quad 5$$

(ii) What is an alternating series? State Leibnitz's test for alternating series. Prove that the series

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \infty$  is a conditionally convergent series. 1+1+3=5

(c) Test the convergence of the series

$$1 + a + a^2 + \dots + a^n + \dots$$

(d) (i) Using Cauchy's condensation test, discuss the convergence of the

series  $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$  5

(ii) Define Cauchy sequence of real numbers. Show that the sequence

$\left\{ \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!} \right\}$  is a

Cauchy sequence. 1+4=5

(e) (i) Show that a convergent sequence of real numbers is a Cauchy sequence. 5

(ii) Using Cauchy's general principle of convergence, show that the

sequence  $\left\{ 1 + \frac{1}{2} + \dots + \frac{1}{n} \right\}$  is not

convergent. 5

(f) (i) Prove that every monotonically increasing sequence which is bounded above converges to its least upper bound. 5

(ii) Show that the limit if exists of a convergent sequence is unique.

5

(g) State and prove  $p$ -series.

(h) (i) Test the convergence of the series

$$x + \frac{3}{5}x^2 + \frac{8}{10}x^3 + \dots + \frac{n^2 - 1}{n^2 + 1}x^n + \dots \quad (x > 0)$$

5

(ii) If  $\{x_n\}$  is a bounded increasing sequence then show that

$$\lim_{n \rightarrow \infty} x_n = \sup\{x_n\}$$

5

(i) (i) Show that a bounded sequence of real numbers has a convergent subsequence.

5

(ii) State and prove Nested Interval theorem.

5

(j) (i) Show that Cauchy sequence of real numbers is bounded.

5

(ii) Test the convergence of the series

$$x^2 + \frac{2^2}{3.4} x^4 + \frac{2^2.4^2}{3.4.5.6} x^6 + \frac{2^2.4^2.6^2}{3.4.5.6.7.8} x^8 + \dots (x > 0)$$

5