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3 (Sem-5/CBCS) MAT HC 1 (N/O)

2022

**MATHEMATICS**

(Honours)

Paper : MAT-HC-5016

(For New Syllabus)

(Complex Analysis)

Full Marks : 60

Time : Three hours

*The figures in the margin indicate full marks for the questions.*

1. Answer **any seven** questions from the following : 1×7=7
- (a) Describe the domain of definition of the function  $f(z) = \frac{z}{z + \bar{z}}$ .
- (b) What is the multiplicative inverse of a non-zero complex number  $z = (x, y)$  ?

Contd.

(c) Verify that  $(3, 1) (3, -1) \left(\frac{1}{5}, \frac{1}{10}\right) = (2, 1)$ .

(d) Determine the accumulation points of the set  $Z_n = \frac{i}{n} (n = 1, 2, 3, \dots)$ .

(e) Write the Cauchy-Riemann equations for a function  $f(z) = u + iv$ .

(f) When a function  $f$  is said to be analytic at a point?

(g) Determine the singular points of the

$$\text{function } f(z) = \frac{2z+1}{z(z^2+1)}.$$

(h)  $\exp(2 \pm 3\pi i)$  is

(i)  $-e^2$

(ii)  $e^2$

(iii)  $2e$

(iv)  $-2e$  (Choose the correct answer)

(i) The value of  $\log(-1)$  is

(i) 0

(ii)  $2n\pi i$

(iii)  $\pi i$

(iv)  $-\pi i$  (Choose the correct answer)

(j) If  $z = x + iy$ , then  $\sin z$  is

(i)  $\sin x \cosh y + i \cos x \sinh y$

(ii)  $-\cos x \cosh y - i \sin x \sinh y$

(iii)  $\cos x \sinh y + i \sin x \cosh y$

(iv)  $\sin x \sinh y - i \cos x \cosh y$

(Choose the correct answer)

(k) If  $\cos z = 0$ , then

(i)  $z = n\pi, (n = 0, \pm 1, \pm 2, \dots)$

(ii)  $z = \frac{\pi}{2} + n\pi, (n = 0, \pm 1, \pm 2, \dots)$

(iii)  $z = 2n\pi, (n = 0, \pm 1, \pm 2, \dots)$

(iv)  $z = \frac{\pi}{2} + 2n\pi, (n = 0, \pm 1, \pm 2, \dots)$

(Choose the correct answer)

(l) If  $z_0$  is a point in the  $z$ -plane, then

$$\lim_{z \rightarrow \infty} f(z) = \infty \text{ if}$$

(i)  $\lim_{z \rightarrow 0} \frac{1}{f(z)} = \infty$

(ii)  $\lim_{z \rightarrow 0} f\left(\frac{1}{z}\right) = 0$

(iii)  $\lim_{z \rightarrow 0} \frac{1}{f(z)} = 0$

(iv)  $\lim_{z \rightarrow 0} f\left(\frac{1}{z}\right) = 0$

(Choose the correct answer)

2. Answer **any four** questions from the following :  
 $2 \times 4 = 8$

(a) Reduce the quantity  $\frac{5i}{(1-i)(2-i)(3-i)}$  to a real number.

(b) Define a connected set and give *one* example.

(c) Find all values of  $z$  such that  $\exp(2z-1) = 1$ .

(d) Show that  $\log(i^3) \neq 3 \log i$ .

(e) Show that

$$2 \sin(z_1 + z_2) \sin(z_1 - z_2) = \cos 2z_2 - \cos 2z_1$$

(f) If  $z_0$  and  $w_0$  are points in the  $z$  plane and  $w$  plane respectively, then prove that  $\lim_{z \rightarrow z_0} f(z) = \infty$  if and only if

$$\lim_{z \rightarrow z_0} \frac{1}{f(z)} = 0.$$

(g) State the Cauchy integral formula. Find

$$\frac{1}{2\pi i} \int_C \frac{1}{z-z_0} dz \text{ if } z_0 \text{ is any point}$$

interior to simple closed contour  $C$ .

(h) Show that  $\int_0^{\frac{\pi}{6}} e^{i2t} dt = \frac{\sqrt{3}}{4} + \frac{i}{4}$ .

3. Answer **any three** questions from the following : 5×3=15

(a) (i) If  $a$  and  $b$  are complex constants, use definition of limit to show that

$$\lim_{z \rightarrow z_0} (az + b) = az_0 + b. \quad 2$$

(ii) Show that

$$\lim_{z \rightarrow 0} \left( \frac{z}{\bar{z}} \right)^2 \text{ does not exist.} \quad 3$$

(b) Suppose that  $\lim_{z \rightarrow z_0} f(z) = w_0$  and

$$\lim_{z \rightarrow z_0} F(z) = W_0.$$

Prove that  $\lim_{z \rightarrow z_0} [f(z)F(z)] = w_0W_0$ .

(c) (i) Show that for the function  $f(z) = \bar{z}$ ,  $f'(z)$  does not exist anywhere. 3

(ii) Show that  $\lim_{z \rightarrow \infty} \frac{4z^2}{(z-1)^2} = 4.$  2

(d) (i) Show that the function  $f(z) = \exp \bar{z}$  is not analytic anywhere. 3

(ii) Find all roots of the equation

$$\log z = i \frac{\pi}{2}. \quad 2$$

(e) If a function  $f$  is analytic at all points interior to and on a simple closed contour  $C$ , then prove that

$$\int_C f(z) dz = 0.$$

(f) Evaluate : 2<sup>1/2</sup>+2<sup>1/2</sup>=5

(i)  $\int_C \frac{e^{-z}}{z - (\pi i/2)} dz$

(ii)  $\int_C \frac{z}{2z+1} dz$

where  $C$  denotes the positively oriented boundary of the square whose sides lie along the lines  $x = \pm 2$  and  $y = \pm 2$ .

(g) Prove that any polynomial

$$P(z) = a_0 + a_1z + a_2z^2 + \dots + a_nz^n \quad (a_n \neq 0)$$

of degree  $n$  ( $n \geq 1$ ) has at least one zero.

(h) Find the Laurent series that represents

$$\text{the function } f(z) = z^2 \sin\left(\frac{1}{z^2}\right) \text{ in the}$$

domain  $0 < |z| < \infty$ .

4. Answer **any three** questions from the following : 10×3=30

(a) (i) If a function  $f$  is continuous throughout a region  $R$  that is both closed and bounded, then prove that there exists a non-negative real number  $\mu$  such that  $|f(z)| \leq \mu$  for all points  $z$  in  $R$ , where equality holds for at least one such  $z$ . 4

(ii) Let a function

$f(z) = u(x, y) + iv(x, y)$  be analytic throughout a given domain  $D$ . If

$|f(z)|$  is constant throughout  $D$ ,

then prove that  $f(z)$  must be constant there too. 3

(iii) Show that the function

$$f(z) = \sin x \cosh y + i \cos x \sinh y$$

is entire. 3

(b) (i) Suppose that  $f(z_0) = g(z_0) = 0$  and that  $f'(z_0)$   $g'(z_0)$  exist, where  $g'(z_0) \neq 0$ . Use definition of derivative to show that

$$\lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}. \quad 3$$

(ii) Show that  $f'(z)$  does not exist at any point if  $f(z) = 2x + ixy^2$ . 3

(iii) If a function  $f$  is analytic at a given point, then prove that its derivatives of all orders are analytic there too. 4

(c) Let the function

$f(z) = u(x, y) + iv(x, y)$  be defined throughout some  $\varepsilon$ -neighbourhood of a point  $z_0 = x_0 + iy_0$ . If  $u_x, u_y, v_x, v_y$  exist everywhere in the neighbourhood, and these partial derivatives are continuous at  $(x_0, y_0)$  and satisfy the Cauchy-Riemann equations at  $(x_0, y_0)$ , then prove that  $f'(z_0)$  exist and  $f'(z_0) = u_x + iv_x$  where the right hand side is to be evaluated at  $(x_0, y_0)$ .

Use it to show that for the function  $f(z) = e^{-x} \cdot e^{-y}$ ,  $f''(z)$  exists everywhere and  $f''(z) = f(z)$ . 6+4=10

(d) (i) Prove that the existence of the derivative of a function at a point implies the continuity of the function at that point.

With the help of an example show that the continuity of a function at a point does not imply the existence of derivative there. 3+5=8

(ii) Find  $f'(z)$  if

$$f(z) = \frac{z-1}{2z+1} \left( z \neq -\frac{1}{2} \right). \quad 2$$

(e) (i) Prove that  $\int_C \frac{dz}{z} = \pi i$  where  $C$  is

the right-hand half  $z = 2e^{i\theta}$

$$\left( -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \right) \text{ of the circle } |z| = 2$$

from  $z = -2i$  to  $z = 2i$ . 5

(ii) If a function  $f$  is analytic everywhere inside and on a simple closed contour  $C$ , taken in the positive sense, then prove that

$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s-z)^2} ds \text{ where } s$$

denotes points on  $C$  and  $z$  is interior to  $C$ . 5

(f) (i) Evaluate  $I = \int_C z^{a-1} dz$

where  $C$  is the positively oriented circle  $z = Re^{i\theta}$  ( $-\pi \leq \theta \leq \pi$ ) about the origin and  $a$  denote any non-zero real number.

If  $a$  is a non-zero integer  $n$ , then what is the value of  $\int_C z^{n-1} dz$ ?

$$4+1=5$$

(ii) Let  $C$  denote a contour of length  $L$ , and suppose that a function  $f(z)$  is piecewise continuous on  $C$ . If  $\mu$  is a non-negative constant such that  $|f(z)| \leq \mu$  for all point  $z$  on  $C$  at which  $f(z)$  is defined, then prove

$$\left| \int_C f(z) dz \right| \leq \mu L.$$

Use it to show that  $\left| \int_C \frac{dz}{z^2-1} \right| \leq \frac{\pi}{3}$  where  $C$  is the arc of the circle  $|z|=2$  from  $z=2$  to  $z=2i$  that lies in the 1st quadrant.

$$3+2=5$$

(g) (i) Apply the Cauchy-Goursat theorem to show that  $\int_C f(z) dz = 0$  when the contour  $C$  is the unit circle  $|z|=1$ , in either direction and  $f(z) = ze^{-z}$ . 4

(ii) If  $C$  is the positively oriented unit circle  $|z|=1$  and  $f(z) = \exp(2z)$  find  $\int_C \frac{f(z)}{z^4} dz$ . 3

(iii) Let  $z_0$  be any point interior to a positively oriented simple closed curve  $C$ . Show that

$$\int_C \frac{dz}{(z-z_0)^{n+1}} = 0, (n=1, 2, \dots). \quad 3$$

(h) (i) Suppose that  $z_n = x_n + iy_n$ , ( $n=1, 2, \dots$ ) and  $z = x + iy$ . Prove that  $\lim_{n \rightarrow \infty} z_n = z$  if and only if

$$\lim_{n \rightarrow \infty} x_n = x \text{ and } \lim_{n \rightarrow \infty} y_n = y. \quad 5$$

(ii) Show that

$$z^2 e^{3z} = \sum_{n=2}^{\infty} \frac{3^{n-2}}{(n-2)!} z^n \quad (|z| < \infty)$$

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