নিৰ্ণায়ক বিটোপ Ch. 4.

আমি মৌলকক্ষৰ লগত সংগতি ৰাখি নিৰ্ণায়ক (determinant)ৰ অলপ কথা শিকিছিলো। এতিয়া নিৰ্ণায়কক সুকীয়াভাৱে অধ্যয়ন কৰা হ'ব।

4.3. Properties of Determinants:-
(নিৰ্ণায়কৰ ধৰ্ম)

ধৰ্ম 1: শাৰী স্তম্ভ আৰু স্তম্ভ শাৰী হিচাপে লিখিলেও নিৰ্ণায়কৰ মান একে থাকে।

ব্যাখ্যা: ধৰা $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

প্ৰথম শাৰী সাপেক্ষে বিস্তৃত কৰি পাম

$$\Delta = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

1 শাৰী আৰু স্তম্ভ নিজৰ মাজত সলনি কৰি পাওঁ

$$\Delta' = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

প্ৰথম স্তম্ভ সাপেক্ষে বিস্তৃত কৰি পাম

$$\Delta' = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

$\therefore \Delta = \Delta'$

Note: মৌলকক্ষৰ হিচাপত চাব গ'লে দেখা যাব যে
$\det A = \det A'$ ($A' = A$ৰ পক্ষান্তৰ মৌলকক্ষ)

Imp
ধৰ্ম 2: এটা নিৰ্ণায়কৰ যিকোনো দুটা শাৰী (অথবা স্তম্ভ) নিজৰ মাজত সালসলনি কৰিলে নিৰ্ণায়কৰ চিনৰ পৰিবৰ্তন হয় (+ টো −, − টো + হয়)

ব্যাখ্যা: $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$\Rightarrow \Delta = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$ ... (1)

এতে প্ৰথম শাৰী (R_1) আৰু তৃতীয় শাৰী (R_3) সালসলনি কৰা হ'ল।

ধৰা হ'ল $\Delta' = \begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix}$ $(R_1 \leftrightarrow R_3)$

বিস্তাৰ কৰিলে পাম (তৃতীয় শাৰী সাপেক্ষে)

$\Delta' = a_1 \begin{vmatrix} b_3 & c_3 \\ b_2 & c_2 \end{vmatrix} - b_1 \begin{vmatrix} a_3 & c_3 \\ a_2 & c_2 \end{vmatrix} + c_1 \begin{vmatrix} a_3 & b_3 \\ a_2 & b_2 \end{vmatrix}$

$= a_1(b_3c_2 - b_2c_3) - b_1(a_3c_2 - a_2c_3) + c_1(a_3b_2 - a_2b_3)$

$= -a_1(b_2c_3 - b_3c_2) + b_1(a_2c_3 - a_3c_2) - c_1(a_2b_3 - a_3b_2)$

$= -[a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)]$

$= -\Delta$. (from (1))

Note: একেদৰে স্তম্ভ 1 (C_1) আৰু স্তম্ভ 3 (C_3) সালসলনি কৰি তৃতীয় স্তম্ভ সাপেক্ষে বিস্তাৰ কৰি আমি পাব পাৰো যে $\Delta' = -\Delta$

ধৰ্ম 3: কোনো নিৰ্ণায়কৰ দুটা শাৰী বা স্তম্ভ একে হ'লে নিৰ্ণায়কৰ মান শূন্য হ'ব।

ধৰা হ'ল $\Delta = D$.

দুটা শাৰী সলনি কৰা হ'ল। তেন্তে

$\Delta' = -D$ (ধৰ্ম 2 মতে)

এতিয়া ~~কিন্তু D = D = ∆~~

$\therefore \Delta = -\Delta$ ($\because$ সলনি কৰা শাৰী দুটা একে)

$\Rightarrow 2\Delta = 0 \Rightarrow \Delta = 0$

(4) নিৰ্ণায়কৰ কোনো শাৰীৰ (বা স্তম্ভৰ) সকলো মৌলক একে উৎপাদকেৰে পূৰণ কৰিলে নিৰ্ণায়কৰ মানক সেই উৎপাদকেৰে পূৰণ কৰা হয়।

$$\begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = k\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(5) কোনো শাৰীৰ (বা স্তম্ভৰ) প্ৰতিটো মৌল দুটা দুটা পদৰ সমষ্টি হ'লে নিৰ্ণায়কটোক দুটা নিৰ্ণায়কৰ সমষ্টি ৰূপে প্ৰকাশ কৰিব পাৰি।

$$\begin{vmatrix} a+\alpha & b+\beta & c+\gamma \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix} = \begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix} + \begin{vmatrix} \alpha & \beta & \gamma \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix}$$

(6) কোনো শাৰীৰ (স্তম্ভৰ) প্ৰতিটো মৌলক একে সংখ্যাৰে পূৰণ কৰি আন এটা শাৰীৰ (স্তম্ভৰ) অনুৰূপ মৌলিকৰ লগত যোগ কৰিলে নিৰ্ণায়কৰ মানৰ সলনি নহয়।

4 $$\begin{vmatrix} a+kb & b & c \\ a'+kb' & b' & c' \\ a''+kb'' & b'' & c'' \end{vmatrix} = \begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix} + \begin{vmatrix} kb & b & c \\ kb' & b' & c' \\ kb'' & b'' & c'' \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix} + k\begin{vmatrix} b & b & c \\ b' & b' & c' \\ b'' & b'' & c'' \end{vmatrix} = \begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix} + 0 = \begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix}$$

Ex $$\begin{vmatrix} 71 & 72 & 73 \\ 70 & 71 & 72 \\ 72 & 73 & 74 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 70 & 71 & 72 \\ 2 & 2 & 2 \end{vmatrix} \quad \begin{matrix} R_1 - R_2 \\ \\ R_3 - R_2 \end{matrix}$$

$$= 2\times\begin{vmatrix} 1 & 1 & 1 \\ 70 & 71 & 72 \\ 1 & 1 & 1 \end{vmatrix} = 2\times 0 = 0 \quad (\because R_1 = R_3)$$

Ex $$\begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix} = \begin{vmatrix} 17\times 6 & 3\times 6 & 6\times 6 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix} = 6\begin{vmatrix} 17 & 3 & 6 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$$

$$= 6\times 0 = 0.$$

ধৰ্ম 6 টাৰ ওপৰত নিৰ্ভৰ কৰি একে ধৰণৰ অংক পৰীক্ষাৰ বাবে প্ৰয়োজন। যদিহে ও অংকবোৰ টো অশান্তিৰে খুঁচি কৰে। Important নাই কিছু অংক কৰিব লাগিব। ~~[illegible]~~ চেষ্টা নকৰিলে পাহৰি যোৱাৰ chance থাকে।

প্ৰশ্ন. যদি x, y, z তিনিওটা বেলেগ বেলেগ ৰাশি আৰু ~~[illegible]~~ যে

$$\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$$

দেখুওৱা যে $1+xyz=0$.

2015, 2017 **প্ৰমাণ**: দেখুৱাব লাগে যে ~~L.H.S.~~ $1+xyz=0$

$$\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix}$$

$$= (-1)^2 \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + xyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$ (প্ৰথম ৰ ত $C_1 \leftrightarrow C_2, C_2 \leftrightarrow C_3$)

$$= (1+xyz) \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$ [4,5 নিয়ম 2 | 4 ২ 5 নিয়ম 4]

$$= (1+xyz) \begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{vmatrix}$$ [$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$ (নিয়ম 6) (−1) ৰে R_1 ক পূৰণ কৰি R_2, R_3 ৰ লগত যোগ দিয়া হৈছে)

$$= (1+xyz)\{(y-x)(z^2-x^2) - (z-x)(y^2-x^2)\} + 0 + 0$$ | 1নং স্তম্ভ সাপেক্ষে প্ৰসাৰণ কৰি

$$= (1+xyz)(y-x)(z-x)\{z-x-y+x\}$$

$$= (1+xyz)(y-x)(z-x)(z-y)$$

প্ৰশ্নমতে, $(1+xyz)(y-x)(z-x)(z-y) = 0$

$\because x \neq y \neq z$, so, $y-x \neq 0$, $z-x \neq 0$ $y-z \neq 0$

$\Rightarrow 1+xyz=0$. Proved.

2014, 2016

Ex. দেখাও যে

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = abc+ab+bc+ca$$

$$L.H.S = abc \begin{vmatrix} \frac{1}{a}+1 & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

$$= abc \begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \\ 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \\ 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \end{vmatrix} \quad R_1 \rightarrow R_1+R_2+R_3$$

$$= abc \begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

6

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \begin{vmatrix} 1 & 0 & 0 \\ \frac{1}{b} & 1 & 0 \\ \frac{1}{c} & 0 & 1 \end{vmatrix} \quad \begin{cases} C_1 \rightarrow \\ C_2 \rightarrow C_2 - C_1 \\ C_3 \rightarrow C_3 - C_1 . \end{cases}$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\{1 \times (1-0)\}$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \text{ Proved.}$$

Ex. $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix}$ - র মান নির্ণয় কর।

$$\Delta = \begin{vmatrix} a-b & b-c & c \\ a^2-b^2 & b^2-c^2 & c^2 \\ bc-ca & ca-ab & ab \end{vmatrix} = (a-b)(b-c)\begin{vmatrix} 1 & 1 & c \\ a+b & b+c & c^2 \\ -c & -a & ab \end{vmatrix}$$

$$= (a-b)(b-c)\begin{vmatrix} 0 & 1 & c \\ a-c & b+c & c^2 \\ a-c & -a & ab \end{vmatrix}$$

$$= (a-b)(b-c)(a-c)\begin{vmatrix} 0 & 1 & c \\ -1 & b+c & c^2 \\ 1 & -a & ab \end{vmatrix}$$

$$= (a-b)(b-c)(a-c)\begin{vmatrix} 0 & 1 & c \\ 0 & a+b+c & c^2-ab \\ -1 & -a & ab \end{vmatrix}$$

$$= -(b-c)(c-a)(a-b)(c^2-ab-ca-bc-c^2)$$

7 $$= (b-c)(c-a)(a-b)(bc+ca+ab).$$

— x —

কিছু অংক Photo মাৰি দিছো। Practice কৰি থাকিবা।

$\Rightarrow \quad \Delta = 0$

EXAMPLE 3 *Show that* $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = 0.$

SOLUTION Let $\Delta = \begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$. Applying $C_2 \to C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} 1 & a+b+c & b+c \\ 1 & b+c+a & c+a \\ 1 & c+a+b & a+b \end{vmatrix}$$

$$\Rightarrow \quad \Delta = (a+b+c) \begin{vmatrix} 1 & 1 & b+c \\ 1 & 1 & c+a \\ 1 & 1 & a+b \end{vmatrix} \qquad \text{[Taking out } a+b+c \text{ common from } C_2]$$

$$\Rightarrow \quad \Delta = (a+b+c) \times 0 = 0 \qquad [\because C_1 \text{ and } C_2 \text{ are identical}]$$

EXAMPLE 4 *Show that* $\begin{vmatrix} b-c & c-a & a-b \\ c-a & a-b & b-c \\ a-b & b-c & c-a \end{vmatrix}$. **[NCERT, CBSE 2009]**

SOLUTION Let $\Delta = \begin{vmatrix} b-c & c-a & a-b \\ c-a & a-b & b-c \\ a-b & b-c & c-a \end{vmatrix}$. Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} 0 & c-a & a-b \\ 0 & a-b & b-c \\ 0 & b-c & c-a \end{vmatrix}$$

$$\Rightarrow \quad \Delta = 0 \qquad [\because C_1 \text{ consists of all zeros}]$$

EXAMPLE 5 *Show that* $\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix} = 0.$ **[NCERT]**

SOLUTION Let $\Delta = \begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix}$. Applying $C_3 \to C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} 1 & bc & ab+bc+ca \\ 1 & ca & ab+bc+ca \\ 1 & ab & ab+bc+ca \end{vmatrix}$$

$\Rightarrow \quad \Delta = (ab+bc+ca)\begin{vmatrix} 1 & bc & 1 \\ 1 & ca & 1 \\ 1 & ab & 1 \end{vmatrix}$ [Taking out $ab+bc+ca$ common from C_3]

$\Rightarrow \quad \Delta = (ab+bc+ca) \times 0 = 0.$ [$\because$ C_1 and C_3 are identical]

EXAMPLE 6 *Without expanding prove that :* $\begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix} = 0.$ [NCERT]

SOLUTION Let $\Delta = \begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$. Applying $R_1 \to R_1 + R_2$, we get

$$\Delta = \begin{vmatrix} x+y+z & x+y+z & x+y+z \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$$

$\Rightarrow \quad \Delta = (x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$ [Taking out $(x+y+z)$ common from R_1]

$\Rightarrow \quad \Delta = (x+y+z) \times 0 = 0$ [$\because$ R_1 and R_3 are identical]

EXAMPLE 7 *Without expanding show that:* $\Delta = \begin{vmatrix} \text{cosec}^2\theta & \cot^2\theta & 1 \\ \cot^2\theta & \text{cosec}^2\theta & -1 \\ 42 & 40 & 2 \end{vmatrix} = 0.$

[NCERT EXEMPLAR]

SOLUTION Applying $C_1 \to C_1 - C_2$, we obtain

$$\Delta = \begin{vmatrix} \text{cosec}^2\theta - \cot^2\theta & \cot^2\theta & 1 \\ \cot^2\theta - \text{cosec}^2\theta & \text{cosec}^2\theta & -1 \\ 42-40 & 40 & 2 \end{vmatrix}$$

$\Rightarrow \quad \Delta = \begin{vmatrix} 1 & \cot^2\theta & 1 \\ -1 & \text{cosec}^2\theta & -1 \\ 2 & 40 & 2 \end{vmatrix} = 0$ [$\because$ C_1 and C_3 are identical]

EXAMPLE 8 *Find the value of the determinant* $\Delta = \begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 8 \\ 6x & 9x & 12x \end{vmatrix}$. [CBSE 2009]

SOLUTION Taking $3x$ common from R_3, we get

$$\Delta = 3x\begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 8 \\ 2 & 3 & 4 \end{vmatrix}$$

$\Rightarrow \quad \Delta = 3x \times 0 = 0$ [$\because$ R_1 and R_3 are identical]

EXAMPLE 9 *Without expanding show that* $\begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix} = 0.$

[NCERT EXEMPLAR, CBSE 2001 C]

SOLUTION Let $\Delta = \begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix}$. Applying $R_1 \to R_1(a)$, $R_2 \to R_2(b)$ and $R_3 \to R_3(c)$, we get

$$\Delta = \frac{1}{abc}\begin{vmatrix} ab^2c^2 & abc & ab+ac \\ bc^2a^2 & abc & bc+ba \\ ca^2b^2 & abc & ac+bc \end{vmatrix} \qquad \left[\because R_1, R_2\, R_3 \text{ are multiplied by } a, b \text{ and } c \text{ respectively, therefore we divide by } abc\right]$$

$$\Rightarrow \quad \Delta = \frac{1}{abc}(abc)^2\begin{vmatrix} bc & 1 & ab+ac \\ ca & 1 & bc+ba \\ ab & 1 & ac+bc \end{vmatrix} \qquad [\text{Taking out } abc \text{ common from } C_1 \text{ and } C_2]$$

$$\Rightarrow \quad \Delta = abc\begin{vmatrix} bc & 1 & ab+bc+ca \\ ca & 1 & ab+bc+ca \\ ab & 1 & ab+bc+ca \end{vmatrix} \qquad [\text{Applying } C_3 \to C_3 + C_1]$$

$$\Rightarrow \quad \Delta = abc\,(ab+bc+ca)\begin{vmatrix} bc & 1 & 1 \\ ca & 1 & 1 \\ ab & 1 & 1 \end{vmatrix} \qquad [\text{Taking out } ab+bc+ca \text{ common from } C_3]$$

$$\Rightarrow \quad \Delta = abc\,(ab+bc+ca) \times 0 = 0 \qquad [\because C_2 \text{ and } C_3 \text{ are identical}]$$

EXAMPLE 10 *Without expanding evaluate the determinant* $\begin{vmatrix} \sin\alpha & \cos\alpha & \sin(\alpha+\delta) \\ \sin\beta & \cos\beta & \sin(\beta+\delta) \\ \sin\gamma & \cos\gamma & \sin(\gamma+\delta) \end{vmatrix}$.

SOLUTION Let $\Delta = \begin{vmatrix} \sin\alpha & \cos\alpha & \sin(\alpha+\delta) \\ \sin\beta & \cos\beta & \sin(\beta+\delta) \\ \sin\gamma & \cos\gamma & \sin(\gamma+\delta) \end{vmatrix}$.

$$\Rightarrow \quad \Delta = \begin{vmatrix} \sin\alpha & \cos\alpha & \sin\alpha\cos\delta + \cos\alpha\sin\delta \\ \sin\beta & \cos\beta & \sin\beta\cos\delta + \cos\beta\sin\delta \\ \sin\gamma & \cos\gamma & \sin\gamma\cos\delta + \cos\gamma\sin\delta \end{vmatrix} \qquad [\because \sin(A+B) = \sin A\cos B + \cos A\sin B]$$

$$\Rightarrow \quad \Delta = \begin{vmatrix} \sin\alpha & \cos\alpha & 0 \\ \sin\beta & \cos\beta & 0 \\ \sin\gamma & \cos\gamma & 0 \end{vmatrix} \qquad [\text{Applying } C_3 \to C_3 - (\cos\delta)\,C_1 - (\sin\delta)\,C_2]$$

$$\Rightarrow \quad \Delta = 0 \qquad [\because C_3 \text{ consists of all zeros}]$$

EXAMPLE 11 *Without expanding evaluate the determinant* $\begin{vmatrix} (a^x+a^{-x})^2 & (a^x-a^{-x})^2 & 1 \\ (a^y+a^{-y})^2 & (a^y-a^{-y})^2 & 1 \\ (a^z+a^{-z})^2 & (a^z-a^{-z})^2 & 1 \end{vmatrix}$, *where* $a, > 0$ *and* $x, y, z \in R$.

SOLUTION Let Δ be the given determinant. Applying $C_1 \to C_1 - C_2$, we get

$$\Delta = \begin{vmatrix} (a^x + a^{-x})^2 - (a^x - a^{-x})^2 & (a^x - a^{-x})^2 & 1 \\ (a^y + a^{-y})^2 - (a^y - a^{-y})^2 & (a^y - a^{-y})^2 & 1 \\ (a^z + a^{-z})^2 - (a^z - a^{-z})^2 & (a^z - a^{-z})^2 & 1 \end{vmatrix}$$

$$\Rightarrow \quad \Delta = \begin{vmatrix} 4 & (a^x - a^{-x})^2 & 1 \\ 4 & (a^y - a^{-y})^2 & 1 \\ 4 & (a^z - a^{-z})^2 & 1 \end{vmatrix} \qquad [\text{Using} : (a+b)^2 - (a-b)^2 = 4ab]$$

$$\Rightarrow \quad \Delta = 4\begin{vmatrix} 1 & (a^x - a^{-x})^2 & 1 \\ 1 & (a^y - a^{-y})^2 & 1 \\ 1 & (a^z - a^{-z})^2 & 1 \end{vmatrix} \qquad [\text{Taking out 4 common from } C_1]$$

$$\Rightarrow \quad \Delta = 4 \times 0 = 0 \qquad [\because C_1 \text{ and } C_3 \text{ are identical}]$$

EXAMPLE 12 *If a, b, c are in A.P., find the value of* $\begin{vmatrix} 2y+4 & 5y+7 & 8y+a \\ 3y+5 & 6y+8 & 9y+b \\ 4y+6 & 7y+9 & 10y+c \end{vmatrix}$. [NCERT]

SOLUTION Applying $R_2 \to R_2\,(2)$, we get

$$\Delta = \frac{1}{2}\begin{vmatrix} 2y+4 & 5y+7 & 8y+a \\ 6y+10 & 12y+16 & 18y+2b \\ 4y+6 & 7y+9 & 10y+c \end{vmatrix}$$

$$\Rightarrow \quad \Delta = \frac{1}{2}\begin{vmatrix} 2y+4 & 5y+7 & 8y+a \\ 0 & 0 & 2b-(a+c) \\ 4y+6 & 7y+9 & 10y+c \end{vmatrix} \qquad \text{Applying } R_2 \to R_2 - (R_1 + R_3)$$

$$\Rightarrow \quad \Delta = \frac{1}{2}\begin{vmatrix} 2y+4 & 5y+7 & 8y+a \\ 0 & 0 & 0 \\ 4y+6 & 7y+9 & 10y+c \end{vmatrix} \qquad [\because a, b, c \text{ are in A.P.} \therefore 2b = a + c]$$

$$\Rightarrow \quad \Delta = 0 \qquad [\because R_2 \text{ consists of zeros only}]$$

REMARK *One can also apply the transformation* $R_1 \to R_1 + R_3 - 2R_2$ *to get the value of* Δ.

EXAMPLE 13 *Without expanding evaluate the determinant* $\Delta = \begin{vmatrix} 265 & 240 & 219 \\ 240 & 225 & 198 \\ 219 & 198 & 181 \end{vmatrix}$.

SOLUTION Applying $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$, we get

$$\Delta = \begin{vmatrix} 46 & 21 & 219 \\ 42 & 27 & 198 \\ 38 & 17 & 181 \end{vmatrix}$$

$$\Rightarrow \quad \Delta = \begin{vmatrix} 4 & 21 & 9 \\ -12 & 27 & -72 \\ 4 & 17 & 11 \end{vmatrix} \qquad [\text{Applying } C_1 \to C_1 - 2C_2 \text{ and } C_3 \to C_3 - 10C_2]$$

$$\Rightarrow \quad \Delta = \begin{vmatrix} 0 & 4 & -2 \\ 0 & 78 & -39 \\ 4 & 17 & 11 \end{vmatrix} \qquad [\text{Applying } R_1 \to R_1 - R_3 \text{ and } R_2 \to R_2 + 3R_3]$$

EXAMPLE 21 *Prove that:* $\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4a^2 b^2 c^2.$

[NCERT, CBSE 2011]

SOLUTION Let $\Delta = \begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix}$. Then,

$$\Delta = abc \begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix}$$ [Taking a, b and c common from R_1, R_2 and R_3 respectively]

$$\Rightarrow \quad \Delta = a^2 b^2 c^2 \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$ [Taking a, b and c common from C_1, C_2 and C_3 respectively]

$$\Rightarrow \quad \Delta = a^2 b^2 c^2 \begin{vmatrix} -1 & 0 & 0 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{vmatrix}$$ [Applying $C_2 \to C_2 + C_1, C_3 \to C_3 + C_1$]

$$\Rightarrow \quad \Delta = a^2 b^2 c^2 \times (-1) \times \begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix}$$ [Expanding along R_1]

$$\Rightarrow \quad \Delta = a^2 b^2 c^2 (-1)(0-4) = 4 a^2 b^2 c^2$$

EXAMPLE 22 *Prove that:* $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix} = xy.$

[NCERT]

SOLUTION Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix}$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we get

$$\Delta = \begin{vmatrix} 1 & 0 & 0 \\ 1 & x & 0 \\ 1 & 0 & y \end{vmatrix}$$

$\Rightarrow \quad \Delta = 1 \times \begin{vmatrix} x & 0 \\ 0 & y \end{vmatrix} - 0 \times \begin{vmatrix} 1 & 0 \\ 1 & y \end{vmatrix} + 0 \times \begin{vmatrix} 1 & x \\ 1 & 0 \end{vmatrix}$ [On expanding along R_1]

$\Rightarrow \quad \Delta = xy$

EXAMPLE 23 *Evaluate:* $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$. [NCERT]

SOLUTION Let Δ be the given determinant. Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$\Delta = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$$

$\Rightarrow \quad \Delta = (b-a)(c-a) \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix}$ [Taking out $(b-a)$ common from R_2 & $(c-a)$ from R_3]

$\Rightarrow \quad \Delta = (b-a)(c-a) \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & c-b \end{vmatrix}$ [Applying $R_3 \to R_3 - R_2$]

$\Rightarrow \quad \Delta = (b-a)(c-a)(c-b) \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & 1 \end{vmatrix}$ [Taking out $(c-b)$ common from R_3]

$\Rightarrow \quad \Delta = (b-a)(c-a)(c-b) \times 1 \times \begin{vmatrix} 1 & b+a \\ 0 & 1 \end{vmatrix}$ [Expanding along C_1]

$\Rightarrow \quad \Delta = (b-a)(c-a)(c-b) \times 1 = (a-b)(b-c)(c-a)$

REMARK *The reader is advised to remember the value of this determinant as a standard result.*

EXAMPLE 24 *Show that:* $\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = xyz(x-y)(y-z)(z-x)$.

[CBSE 2000, 2010 C, 2011]

SOLUTION Let $\Delta = \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix}$. Taking x, y and z common from C_1, C_2 and C_3 respectively, we get

$$\Delta = xyz \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$\Rightarrow \quad \Delta = xyz\begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ x^2 & y^2-x^2 & z^2-x^2 \end{vmatrix}$ [Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$]

$\Rightarrow \quad \Delta = xyz\,(y-x)\,(z-x)\begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 1 \\ x^2 & y+x & z+x \end{vmatrix}$ [Taking $(y-x)$ and $(z-x)$ common from C_2 from C_3 respectively.]

$\Rightarrow \quad \Delta = xyz\,(y-x)\,(z-x) \times 1 \times \begin{vmatrix} 1 & 1 \\ y+x & z+x \end{vmatrix}$ [Expanding along R_1]

$\Rightarrow \quad \Delta = xyz\,(y-x)\,(z-x)\,(z+x-y-x)$

$\Rightarrow \quad \Delta = xyz\,(x-y)\,(y-z)\,(z-x)$

EXAMPLE 25 *Prove that:* $\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \beta+\gamma & \gamma+\alpha & \alpha+\beta \end{vmatrix} = (\alpha-\beta)\,(\beta-\gamma)\,(\gamma-\alpha)\,(\alpha+\beta+\gamma).$

[NCERT, CBSE 2007C, 2008, 2010 C]

SOLUTION Let $\Delta = \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \beta+\gamma & \gamma+\alpha & \alpha+\beta \end{vmatrix}$. Applying $R_3 \to R_1 + R_3$, we get

$$\Delta = \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \alpha+\beta+\gamma & \alpha+\beta+\gamma & \alpha+\beta+\gamma \end{vmatrix}$$

$\Rightarrow \quad \Delta = (\alpha+\beta+\gamma)\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ 1 & 1 & 1 \end{vmatrix}$ [Taking out $(\alpha+\beta+\gamma)$ common from R_3]

$\Rightarrow \quad \Delta = (\alpha+\beta+\gamma)\begin{vmatrix} \alpha & \beta-\alpha & \gamma-\alpha \\ \alpha^2 & \beta^2-\alpha^2 & \gamma^2-\alpha^2 \\ 1 & 0 & 0 \end{vmatrix}$ [Applying $C_1 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$]

$\Rightarrow \quad \Delta = (\alpha+\beta+\gamma)\,(\beta-\alpha)\,(\gamma-\alpha)\begin{vmatrix} \alpha & 1 & 1 \\ \alpha^2 & \beta+\alpha & \gamma+\alpha \\ 1 & 0 & 0 \end{vmatrix}$ [Taking $(\beta-\alpha)$ common from C_2 and $(\gamma-\alpha)$ from C_3]

$\Rightarrow \quad \Delta = (\alpha+\beta+\gamma)\,(\beta-\alpha)\,(\gamma-\alpha) \times 1 \times \begin{vmatrix} 1 & 1 \\ \beta+\alpha & \gamma+\alpha \end{vmatrix}$ [Expanding along R_3]

$\Rightarrow \quad \Delta = (\alpha+\beta+\gamma)\,(\beta-\alpha)\,(\gamma-\alpha)\,(\gamma+\alpha-\beta-\alpha) = (\alpha-\beta)\,(\beta-\gamma)\,(\gamma-\alpha)\,(\alpha+\beta+\gamma).$

EXAMPLE 26 *In a* ΔABC, *if* $\begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A+\sin^2 A & \sin B+\sin^2 B & \sin C+\sin^2 C \end{vmatrix} = 0$, *then prove that* ΔABC *is an isosceles triangle.*

[NCERT EXEMPLAR]

SOLUTION Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A+\sin^2 A & \sin B+\sin^2 B & \sin C+\sin^2 C \end{vmatrix}$. Then,

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ \sin A & \sin B & \sin C \\ \sin A + \sin^2 A & \sin B + \sin^2 B & \sin C + \sin^2 C \end{vmatrix} \quad [\text{Applying } R_2 \to R_2 - R_1]$$

$\Rightarrow$
$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ \sin A & \sin B & \sin C \\ \sin^2 A & \sin^2 B & \sin^2 C \end{vmatrix} \quad [\text{Applying } R_3 \to R_3 - R_2]$$

$\Rightarrow$
$$\Delta = \begin{vmatrix} 1 & 0 & 0 \\ \sin A & \sin B - \sin A & \sin C - \sin A \\ \sin^2 A & \sin^2 B - \sin^2 A & \sin^2 C - \sin^2 A \end{vmatrix} \quad [\text{Applying } C_2 \to C_2 - C_1, C_3 \to C_3 - C_1]$$

$\Rightarrow$
$$\Delta = (\sin B - \sin A)(\sin C - \sin A) \begin{vmatrix} 1 & 0 & 0 \\ \sin A & 1 & 1 \\ \sin^2 A & \sin B + \sin A & \sin C + \sin A \end{vmatrix}$$

[Taking $\sin B - \sin A$ common from C_2 and $\sin C - \sin A$ from C_3]

$\Rightarrow$ $\Delta = (\sin B - \sin A)(\sin C - \sin A)\{(\sin C + \sin A) - (\sin B + \sin A)\}$ [Expanding along R_1]

$\Rightarrow$ $\Delta = (\sin B - \sin A)(\sin C - \sin A)(\sin C - \sin B)$

Now, $\Delta = 0$

$\Rightarrow$ $(\sin B - \sin A)(\sin C - \sin A)(\sin C - \sin B) = 0$

$\Rightarrow$ either $\sin B - \sin A = 0$ or, $\sin C - \sin A = 0$ or, $\sin C - \sin B = 0$

$\Rightarrow$ either $\sin A - \sin B = 0$ or, $\sin C = \sin A = 0$ or, $\sin C - \sin B = 0$

$\Rightarrow$ $A = B$ or $C = A$ or $B = C$

$\Rightarrow$ $BC = CA$ or, $AB = BC$ or $CA = AB$

$\Rightarrow$ ΔABC is isosceles

EXAMPLE 27 *In a ΔABC, if* $\begin{vmatrix} 1 & 1 & 1 \\ 1 + \cos A & 1 + \cos B & 1 + \cos C \\ \cos^2 A + \cos A & \cos^2 B + \cos B & \cos^2 C + \cos C \end{vmatrix} = 0$, *show that ΔABC is an isosceles.* **[NCERT EXEMPLAR, CBSE 2016]**

SOLUTION Proceed as in Example 26.

EXAMPLE 28 *Prove that:* $\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = (a - b)(b - c)(c - a)(a + b + c)$

[NCERT, CBSE 2011, 2012, 2013]

SOLUTION Let $\Delta = \begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}$. Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we obtain

$$\Delta = \begin{vmatrix} 1 & a & a^3 \\ 0 & b - a & b^3 - a^3 \\ 0 & c - a & c^3 - a^3 \end{vmatrix}$$

$\Rightarrow$
$$\Delta = (b - a)(c - a) \begin{vmatrix} 1 & a & a^3 \\ 0 & 1 & b^2 + a^2 + ab \\ 0 & 1 & c^2 + a^2 + ac \end{vmatrix} \quad [\text{Taking out } (b - a) \text{ from } R_2 \text{ and } (c - a) \text{ from } R_3]$$

$$\Rightarrow \quad \Delta = (b-a)(c-a)\begin{vmatrix} 1 & a & a^3 \\ 0 & 0 & (b^2-c^2)+(ab-ac) \\ 0 & 1 & c^2+a^2+ac \end{vmatrix} \qquad \text{[Applying } R_2 \to R_2 - R_3\text{]}$$

$$\Rightarrow \quad \Delta = (b-a)(c-a)\begin{vmatrix} 1 & a & a^3 \\ 0 & 0 & (b-c)(b+c+a) \\ 0 & 1 & c^2+a^2+ac \end{vmatrix}$$

$$\Rightarrow \quad \Delta = (b-a)(c-a)(b-c)\begin{vmatrix} 1 & a & a^3 \\ 0 & 0 & a+b+c \\ 0 & 1 & c^2+a^2+ac \end{vmatrix} \qquad \text{[Taking out } (b-c) \text{ common from } R_2\text{]}$$

$$\Rightarrow \quad \Delta = (b-a)(c-a)(b-c) \times 1 \times \begin{vmatrix} 0 & a+b+c \\ 1 & c^2+a^2+ac \end{vmatrix} \qquad \text{[Expanding along } C_1\text{]}$$

$$\Rightarrow \quad \Delta = (b-a)(c-a)(b-c)\{0-(a+b+c)\} = (a-b)(b-c)(c-a)(a+b+c).$$

EXAMPLE 29 *Show that* $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca).$

[NCERT, CBSE 2007, 2011, 2013, 2014]

SOLUTION Let $\Delta = \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix}$. Multiplying C_1, C_2 and C_3 by a, b and c respectively, we get

$$\Delta = \frac{1}{abc}\begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \\ abc & abc & abc \end{vmatrix}$$

$$\Rightarrow \quad \Delta = \frac{abc}{abc}\begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \\ 1 & 1 & 1 \end{vmatrix} \qquad \text{[Taking } abc \text{ common from } R_3\text{]}$$

$$\Rightarrow \quad \Delta = -\begin{vmatrix} a^2 & b^2 & c^2 \\ 1 & 1 & 1 \\ a^3 & b^3 & c^3 \end{vmatrix} \qquad \text{[Applying } R_2 \leftrightarrow R_3\text{]}$$

$$\Rightarrow \quad \Delta = \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} \qquad \text{[Applying } R_1 \leftrightarrow R_2\text{]}$$

$$\Rightarrow \quad \Delta = \begin{vmatrix} 1 & 0 & 0 \\ a^2 & b^2-a^2 & c^2-a^2 \\ a^3 & b^3-a^3 & c^3-a^3 \end{vmatrix} \qquad \text{[Applying } C_2 \to C_2 - C_1 \text{ and } C_3 \to C_3 - C_1\text{]}$$

$$\Rightarrow \quad \Delta = \begin{vmatrix} 1 & 0 & 0 \\ a^2 & (b-a)(b+a) & (c-a)(c+a) \\ a^3 & (b-a)(b^2+ba+a^2) & (c-a)(c^2+ca+a^2) \end{vmatrix}$$

$$\Rightarrow \quad \Delta = (b-a)(c-a)\begin{vmatrix} 1 & 0 & 0 \\ a^2 & b+a & c+a \\ a^3 & b^2+a^2+ab & c^2+ac+a^2 \end{vmatrix}$$ $\left[\begin{array}{l}\text{Taking } (b-a) \text{ and } (c-a) \text{ common} \\ \text{from } C_2 \text{ and } C_3 \text{ respectively}\end{array}\right]$

$$\Rightarrow \quad \Delta = (b-a)(c-a)\times 1\times\begin{vmatrix} b+a & c+a \\ b^2+a^2+ab & c^2+a^2+ac \end{vmatrix}$$ [Expanding along R_1]

$$\Rightarrow \quad \Delta = (b-a)(c-a)\begin{vmatrix} b-c & c+a \\ b^2-c^2+ab-ac & c^2+a^2+ac \end{vmatrix}$$ [Applying $C_1 \to C_1 - C_2$]

$$\Rightarrow \quad \Delta = (b-a)(c-a)\begin{vmatrix} b-c & c+a \\ (b^2-c^2)+a(b-c) & c^2+a^2+ac \end{vmatrix}$$

$$\Rightarrow \quad \Delta = (b-a)(c-a)(b-c)\begin{vmatrix} 1 & c+a \\ b+c+a & c^2+a^2+ac \end{vmatrix}$$ [Taking $(b-c)$ common from C_1]

$$\Rightarrow \quad \Delta = (b-a)(c-a)(b-c)(c^2+a^2+ac-bc-c^2-ac-ab-ac-a^2)$$

$$\Rightarrow \quad \Delta = (b-a)(c-a)(b-c)(-bc-ab-ac) = (a-b)(b-c)(c-a)(ab+bc+ca).$$

EXAMPLE 30 *If* $x \neq y \neq z$ *and* $\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ \ldots \end{vmatrix} = 0$, *then prove that* $xyz = -1$.

EXAMPLE 31 *For any scalar p prove that* $\Delta = \begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix} = (1+pxyz)(x-y)(y-z)(z-x)$.

[NCERT, CBSE 2010]

SOLUTION **We have,**

$$\Delta = \begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix}$$

$$\Rightarrow \quad \Delta = \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & px^3 \\ y & y^2 & py^3 \\ z & z^2 & pz^3 \end{vmatrix} \quad \left[\because \text{Each element in III column is sum of two elements}\right]$$

$$\Rightarrow \quad \Delta = -\begin{vmatrix} 1 & x^2 & x \\ 1 & y^2 & y \\ 1 & z^2 & z \end{vmatrix} + pxyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

[Interchanging C_1 and C_3 in first det. Taking x, y, z common from R_1, R_2, R_3 respectively and p from C_3 in 2nd det.]

$$\Rightarrow \quad \Delta = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + pxyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \quad [\text{Interchanging } C_2 \text{ and } C_3 \text{ in first determinant}]$$

$$\Rightarrow \quad \Delta = (1+pxyz) \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$\Rightarrow \quad \Delta = (1 + pxyz)\begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{vmatrix}$$ [Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$]

$$\Rightarrow \quad \Delta = (1 + pxyz)(y-x)(z-x)\begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 1 & z+x \end{vmatrix}$$ [Taking $(y-x)$ and $(z-x)$ common from R_2 and R_3 respectively]

$$\Rightarrow \quad \Delta = (1 + pxyz)(y-x)(z-x)\begin{vmatrix} 1 & y+x \\ 1 & z+x \end{vmatrix}$$ [Expanding along C_1]

$$\Rightarrow \quad \Delta = (1 + pxyz)(y-x)(z-x)(z+x-y-x) = (1+pxyz)(x-y)(y-z)(z-x)$$

$$\begin{vmatrix} 1 & a & a^2 - bc \end{vmatrix}$$