(*ii*) In Experiment 2, relative motion between coil C and current carrying coil C', again changes the magnetic flux linked with coil C. In either case, an e.m.f. is induced in coil C. It is this induced e.m.f. which causes electric current in the coil and galvanometer shows some deflection.

(*iii*) In Experiment 3, when the tapping key K is pressed, the current in coil C' increases from zero to maximum. Therefore, resulting magnetic field of coil C' increases from zero to maximum. Consequently, the magnetic flux linked with the neighbouring coil C also increases. It is this change in magnetic flux linked with coil C that induces e.m.f. in coil C.

When key K is kept pressed current in coil C' is constant. Therefore, there is *no change in magnetic flux linked with coil C. Hence, current induced in coil C drops to zero.*

When key K is released, current in coil C' decreases from maximum to zero. The resulting magnetic field of coil C' decreases from maximum to zero in a short time. This results in a decrease in magnetic flux linked with coil C. Hence an e.m.f. is induced in coil C in the opposite direction.

All experimental observations lead us to conclude that induced e.m.f. appears in a coil whenever the amount of magnetic flux linked with the coil changes. ***Note that mere presence of magnetic flux is not enough. The amount of magnetic flux linked with the coil must change in order to produce any induced e.m.f. in the coil.***

4(a).5. FARADAY'S LAWS OF ELECTROMAGNETIC INDUCTION

Following are the two laws of electromagnetic induction as given by Faraday. Both the laws follow from Faraday's experiments dicussed above.

First Law

Whenever the amount of magnetic flux linked with a circuit changes, an e.m.f. is induced in the circuit. The induced e.m.f. lasts so long as the change in magnetic flux continues.

Second Law

The magnitude of e.m.f. induced in a circuit is directly proportional to the rate of change of magnetic flux linked with the circuit.

Explanation

First Law. In Faraday's experiment, when magnet is moved towards the coil, number of magnetic lines of force linked with the coil increases, *i.e.*, magnetic flux increases. When the magnet is moved away, the magnetic flux linked with the coil decreases. In both the cases, galvanometer shows deflection indicating that e.m.f. is induced in the coil.

When there is *no relative motion* between the magnet and the coil, magnetic flux linked with the coil remains constant. That is why galvanometer shows no deflection. Thus, induced e.m.f. is produced when magnetic flux changes and induced e.m.f. continues so long as the change in magnetic flux continues. This is first law. The same results follow from Faraday's second experiment.

Second Law. In Faraday's experiment, when magnet is moved ***faster***, the magnetic flux linked with the coil changes at a faster rate. Therefore, galvanometer deflection is ***more.*** However, when the magnet is moved slowly, rate of change of magnetic flux is smaller. Therefore, galvanometer deflection is smaller, *i.e.*, induced e.m.f. is smaller. Hence magnitude of e.m.f induced varies directly as the rate of change of magnetic flux linked with the coil. This is second law.

If ϕ_1 is amount of magnetic flux linked with a coil at any time and ϕ_2 is the magnetic flux linked with the coil after t sec., then

$$\text{Rate of change of magnetic flux} = \frac{\phi_2 - \phi_1}{t}$$

According to Faraday's second law, induced e.m.f.

$$e \propto \frac{(\phi_2 - \phi_1)}{t} \quad \text{or} \quad e = \frac{k(\phi_2 - \phi_1)}{t} \qquad \text{...(6)}$$

where k is a constant of proportionality.

As $k = 1$ (in all systems of units)

$$\therefore \quad \boxed{e = \frac{\phi_2 - \phi_1}{t}} \quad ...(7)$$

If $d\,\phi$ is small change in magnetic flux in a small time dt, we can rewrite (7) as

$$\boxed{e = \frac{-d\phi}{dt}} \quad ...(8)$$

Negative sign is taken because induced e.m.f. always opposes any change in magnetic flux associated with the circuit. This follows from Lenz's law discussed in Art. 4(a).6.

In case of a closely wound coil of N turns, change in magnetic flux associated with each turn is the same. Therefore, total induced e.m.f. is given by $e = -N\frac{d\phi}{dt}$

By increasing number of turns N in the coil, we can increase the induced e.m.f.

Sample Problem **The magnetic flux threading a coil changes from 12×10^{-3} Wb to 6×10^{-3} Wb in 0·01 s. Calculate the induced e.m.f.**

Sol. Here, $\phi_1 = 12 \times 10^{-3}$ Wb, $\phi_2 = 6 \times 10^{-3}$ Wb, $dt = 0{\cdot}01$ s $= 10^{-2}$ s, $e = ?$

$$e = \frac{-d\phi}{dt} = \frac{-(\phi_2 - \phi_1)}{dt} = \frac{-(6\times10^{-3} - 12\times10^{-3})}{10^{-2}} = 0{\cdot}6 \text{ volt.}$$

4(a).6. LENZ'S LAW

This law gives us the direction of current induced in a circuit.

According to Lenz's law, the polarity of the induced e.m.f. is such that it opposes the change in magnetic flux responsible for its production.

For example, in Fig. 4(a).7, when north pole of a bar magnet is being pushed towards the coil, the amount of magnetic flux linked with the coil increases. Current is induced in the coil in such a direction that it *opposes the increase in flux*. This is possible only when current induced in the coil is in *anticlockwise direction* with respect to an observer on the side of the bar magnet. The magnetic moment $\vec{M}$ associated with this induced current has north polarity towards the north pole of the approaching bar magnet, as shown in Fig. 4(a).7.

Similarly, when north pole of the bar magnet is moved away from the coil, Fig. 4(a).8, the magnetic flux linked with the coil decreases. To counter this decrease in magnetic flux, current induced in the coil is in clockwise direction so that its south pole faces the receding north pole of the bar magnet.

FIGURE 4(a).7

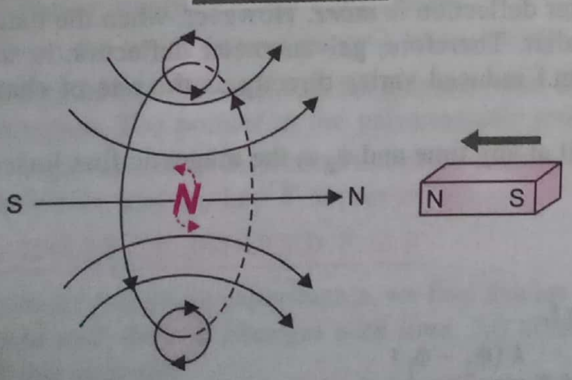
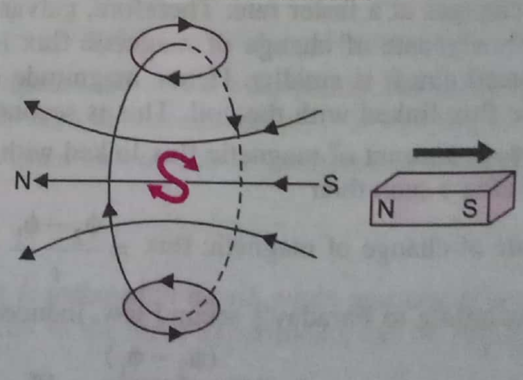


FIGURE 4(a).8



This would result in an attractive force, which opposes the motion of the magnet and the corresponding decrease in magnetic flux.

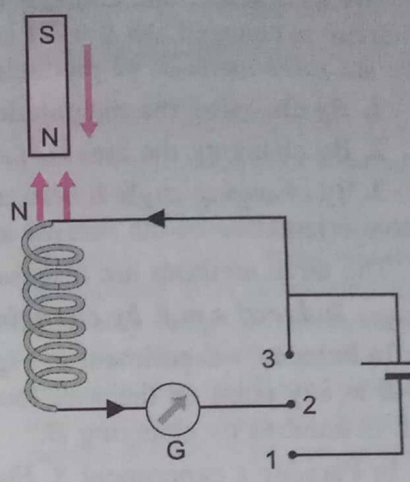
Experimental Verification of Lenz's Law

Figure 4(a). 9 shows the experimental set up for verifying Lenz's law. A coil of a few turns is connected to a cell *C* and a sensitive galvanometer *G* through a two way key 1, 2, 3.

Put in the plug of key between 1 and 2. Cell sends current through the coil. At the upper face of the coil, the current is *anticlockwise*, which would produce *north pole* on this face. Suppose the galvanometer deflection is to the *right*. Obviously, if galvanometer deflection were to the left, current would be clockwise at the upper face, which would behave as south pole.

Remove the plug of key from 1 and 2. Insert the plug of key between 2 and 3. Now, move *N*-pole of a bar magnet towards the coil. The galvanometer shows a sudden deflection to the right indicating that current induced in the coil is anticlockwise and upper end of the coil behaves as north. It opposes the inward motion of *N*-pole of the bar magnet, which is the cause of induced current.

FIGURE 4(a).9



Similarly, when *N*-pole of the bar magnet is moved away from the coil, the galvanometer shows a sudden deflection to the *left*, indicating that current induced in the coil is *clockwise* and upper end of the coil behaves as south. It opposes the outward motion of *N*-pole of the bar magnet., i.e., cause of induced e.m.f. is opposed.

Exactly similar results follow when *S*-pole of magnet is moved instead of *N*-pole.

Hence *induced current always opposes the change in magnetic flux which produces it.* This verifies Lenz's law.

4(a).7. LENZ'S LAW AND ENERGY CONSERVATION

Lenz's law is in accordance with the law of conservation of energy.

For example, in the experimental verification of Lenz's law, when *N*-pole of magnet is moved towads the coil, the upper face of the coil acquires north polarity. Therefore, work has to be done **against** the force of repulsion, in bringing the magnet closer to the coil. Similarly, when *N* pole of magnet is moved away, south polarity developes on the upper face of the coil. Therefore, work has to be done against the force of attraction, in taking the magnet away from the coil.

It is this mechanical work done in moving the magnet w.r.t. the coil that changes into electrical energy producing induced current. Thus, energy is being transformed only.

When we do not move the magnet, work done is zero. Therefore, induced current is also not produced.

Hence Lenz's law obeys the principle of energy conservation.

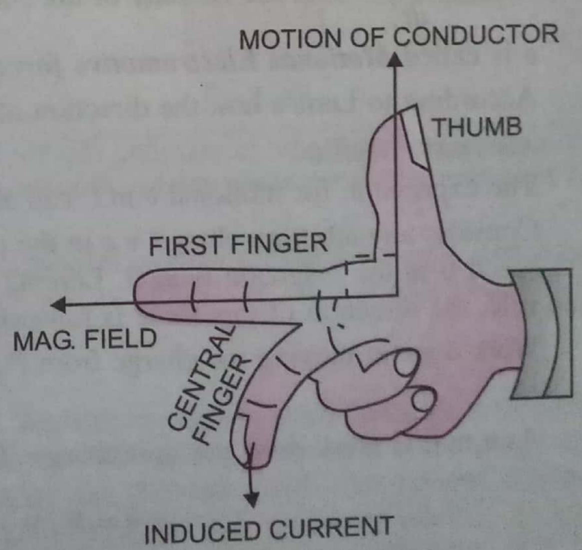
Conversely, Lenz's law can be treated as a consequence of the principle of energy conservation.

4(a).8. FLEMING'S RIGHT HAND RULE

Fleming's right hand rule also gives us the direction of induced e.m.f./current, in a conductor moving in a magnetic field. According to this rule,

If we stretch the first finger, central finger and thumb of our right hand in mutually perpendicular directions such that first finger points along the direction of the field and thumb is along the direction of motion of the conductor, then the central finger would give us the direction of induced current, **Fig 4(a). 10.**

FIGURE 4(a).10



The direction of induced current given by **Lenz's law** and **Fleming's right hand rule** will obviously be the same.

4

SELF AND MUTUAL INDUCTANCE : THEIR MEASUREMENTS : GROWTH AND DECAY OF CURRENTS IN DC CIRCUITS

Resume of basic concepts. *Faraday's law of electromagnetic induction :*

In a varying magnetic field, an emf is induced in any closed circuit which is equal to the negative of the time rate of change of the magnetic flux through the circuit, thus

$$e = -\frac{d\varphi}{dt}.$$

Magnetic flux through a given surface is

$$\varphi = \Sigma \vec{B} \cdot \Delta \vec{s} \quad \text{or} \quad \int \vec{B} \cdot d\vec{s}.$$

4.1. Self and Mutual Inductance

(a) *Self-inductance.* Suppose we have a coil A and pass a current through it. A constant flux is linked up with the coil. If the current is changed by some means, there will be a change of magnetic flux linked with the coil. Due to this change of its own flux an emf and hence a current is produced in the coil itself. The emf so produced is called a self-induced emf, and the corresponding current is called the self-induced current. This phenomenon of production of a self-induced emf and hence a current in a coil due to a change of its own flux is called self-induction.

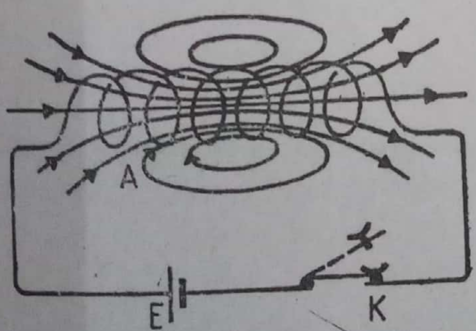


Fig. 4.1

Suppose i is the instantaneous value of a current through a coil. The magnetic induction field is proportional to the current whatever be the shape of the conductor and so flux linked with the coil is also proportional to the current. That is,

$$\varphi \propto i \quad \text{and} \quad \frac{d\varphi}{dt} \propto \frac{di}{dt}.$$

By the law of electromagnetic induction, $e = -\frac{d\varphi}{dt}$

$$\therefore \qquad e \propto -\frac{di}{dt}$$

$$\text{or} \qquad e = -L\frac{di}{dt}$$

where L is a constant of the coil depending on its size, shape, number of turns and the material of the core. This is called the coefficient of self-inductance of the coil.

Definition. Let us put $\frac{di}{dt} = 1$ in the above formula introducing self-inductance. We have $e = -L$. Thus, the coefficient of self-induction or simply self-inductance may be *defined as the self-induced emf produced by unit rate of change of current.*

Unit of self-inductance (L): According to the introductory formula, the unit of self-inductance is $VA^{-1}s$ called a henry (H). Hence a *henry is the unit of self-inductance* and *equals the self-inductance of a coil in which one volt emf is induced by unit rate of change of current,* that is, *one ampere per second.*

We may also define self-inductance and its unit in two other alternative ways based on this introductory formula. We have

$$e = -L\frac{di}{dt}.$$

But e is also given by, $e = -\frac{d\varphi}{dt}$ by law of electromagnetic induction.

$$\therefore \qquad L\frac{di}{dt} = \frac{d\varphi}{dt}$$

$$L\,di = d\varphi$$

or $\int L\,di = \int d\varphi + \varphi_0$ (a constant) or $Li = \varphi + \varphi_0$.

or When $i = 0$, $\varphi = 0$.

$$0 = 0 + \varphi_0 \quad \text{or} \quad \varphi_0 = 0.$$

$$\therefore \qquad \varphi = Li.$$

$\therefore$ Put $i = 1$ ampere, then $\varphi = L$. Thus self-inductance may be defined *as the flux linked with the coil, when there is 1 ampere current through it.*

Since i is the instantaneous current and $-L\frac{di}{dt}$ is the instantaneous induced emf, the instantaneous rate of doing work in the circuit will be $\left(-L\frac{di}{dt}\right)i$, because rate of doing work is given by the product of emf and current (Power = volt × ampere). Therefore, the total work done in establishing the final steady current in a short time τ is given by

$$W=\int_0^\tau\left(-L\frac{di}{dt}\right)dt=-L\int_0^I i\,di=-\tfrac{1}{2}LI^2.$$

Thus, self-inductance may also *be defined as twice the work done in establishing unit steady current, that is, 1 ampere current through it.*

(b) *Mutual Inductance : Coefficient of Coupling.* If we have two coils P ans S placed close to each other, then on changing the current through one of them, say, P by some means, an emf and hence a current is induced in S if S is closed. This phenomenon of production of an induced emf in a coil due to a change of current in another coil is called mutual induction.

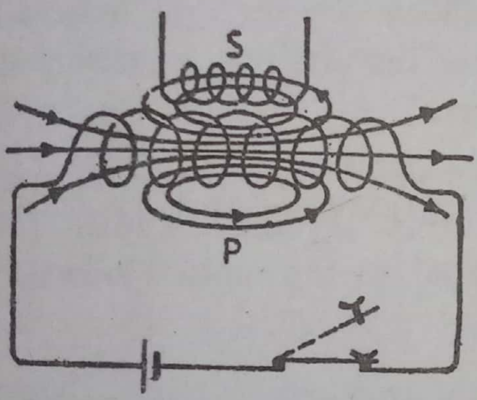


Fig. 4.2

If i is the instantaneous current through the primary coil P, then the instantaneous flux with the secondary coil will be proportional to this instantaneous primary current. That is

$$\varphi_s \propto i \text{ and } \frac{d\varphi_s}{dt}\propto\frac{di}{dt}.$$

But the law of electromagnetic induction, $e_s=-\frac{d\varphi_s}{dt}$

$$\therefore \qquad e_s \propto -\frac{di}{dt}$$

or

$$e_s=-M\frac{di}{dt}$$

where M is a constant called coefficient of mutual induction between the coils.

Definition. Putting $\frac{di}{dt}=1$, we have $e_s=-M$. Thus the coefficient of mutual induction may be defined as the mutual induced emf produced by unit rate of change of current in the other coil.

Alternatively we have,

$$e_s=-M\frac{di}{dt}=-\frac{d\varphi_s}{dt}$$

or $$Mdi=d\varphi_s \quad \text{or} \quad \varphi_s=Mi.$$

Thus mutual inductance may also be defined as *the flux linked with the secondary coil when there is one ampere current through the primary coil.*

4.2. Reciprocity Theorem : Coefficient of Coupling

(a) *Reciprocity Theorem.* The coefficient of mutual inductance between two coils is the same whether the first one is primary and the second one is secondary or vice versa

Let us designate the two circuits as C_1 and C_2.

Let i_1 and i_2 be the instantaneous current through 1 and 2. Then the flux (φ_{12}) through 2 due to 1 is related to the mutual inductance of 2 relative to 1 (M_{12}) by

$$M_{12}=\frac{\varphi_{12}}{i_1} \text{ and similarly } M_{21}=\frac{\varphi_{21}}{i_2}$$

Suppose P is any point on the circuit 2 at a distance r_{12} from an element of current $i_1\vec{dl_1}$ of circuit 1. Then, according to Biot-Savart law, the magnetic field at P due to $i_1\vec{dl_1}$ is $\frac{\mu_0}{4\pi}\frac{i_1\vec{dl_1}\times\vec{r_{12}}}{r_{12}^3}$ and therefore, the total field at P due to 1 is

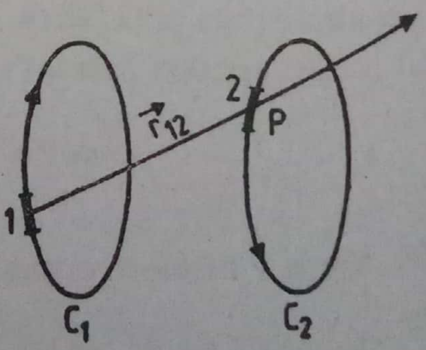


Fig. 4.3

$$B_1=\frac{\mu_0}{4\pi}\oint_{C_1}\frac{i_1\vec{dl_1}\times\vec{r_{12}}}{r_{12}^3}$$

where $\oint_{C_1}$ stands for integration along the entire path of circuit 1.

Using the well-known result $\vec{\nabla}\left(\frac{1}{r}\right)=-\frac{\vec{r}}{r^3}$ and remembering that i_1 is a constant with respect to the integral we have

$$B_1=\frac{\mu_0 i_1}{4\pi}\oint_{C_1}\vec{dl}_1\times\left[-\vec{\nabla}\left(\frac{1}{r_{12}}\right)\right]$$

$$=\frac{\mu_0 i_1}{4\pi}\oint_{C_1}\vec{\nabla}\left(\frac{1}{r_{12}}\right)\times\vec{dl}_1$$

$$(\because\ \vec{a}\times\vec{b}=-\vec{b}\times\vec{a})$$

Using the identity. $\vec{\nabla}\times\psi\vec{A}=\psi\vec{\nabla}\times\vec{A}+\vec{\nabla}\psi\times\vec{A}$

or $\vec{\nabla}\psi\times\vec{A}=\vec{\nabla}\times\psi\vec{A}-\psi\vec{\nabla}\times\vec{A}$

for integration by parts we have

$$B_1=\frac{\mu_0 i_1}{4\pi}\oint_{C_1}\vec{\nabla}\times\frac{\vec{dl}_1}{r_{12}}-\frac{\mu_0 i_1}{4\pi}\oint_{C_1}\frac{1}{r_{12}}\vec{\nabla}\times\vec{dl}_1$$

Now, $\vec{\nabla}\times\vec{dl}_1$ is zero because $\vec{\nabla}$ is the operator with respect to field points, that is points on circuit 2.

$$\therefore\quad B_1=\frac{\mu_0 i_1}{4\pi}\oint_{C_1}\vec{\nabla}\times\frac{\vec{dl}_1}{r_{12}}$$

Now, the flux (φ_{12}) linked with C_2 due to 1 is given by

$$\varphi_{12}=\int_{S_2}\vec{B}_1.\vec{ds}_2=\iint_{S_2}\left[\frac{\mu_0 i_1}{4\pi}\oint_{C_1}\vec{\nabla}\times\frac{\vec{dl}_1}{r_{12}}\right].\vec{dS}_2$$

$$=\frac{\mu_0 i_1}{4\pi}\iint_{S_2}\oint_{C_1}\left(\vec{\nabla}\times\frac{\vec{dl}_1}{r_{12}}\right).\vec{dS}_2$$

Since interchange of integrals is permissible we can write

$$\varphi_{12}=\frac{\mu_0 i_1}{4\pi}\oint_{C_1}\int_{S_2}\left(\vec{\nabla}\times\frac{\vec{dl}_1}{r_{12}}\right).\vec{dS}_2$$

using Stoke's theorem to convert the surface integral into line integral we have

$$\varphi_{12}=\frac{\mu_0 i_1}{4\pi}\oint_{C_1}\oint_{C_2}\frac{\vec{dl}_1.\vec{dl}_2}{r_{12}}$$

$$\therefore \quad M_{12}=\frac{\varphi_{12}}{i_1}=\frac{\mu_0}{4\pi}\oint_{C_1}\oint_{C_2}\frac{\vec{dl_1}.\vec{dl_2}}{r_{12}},$$

which is known as the Neumann equation. As it is symmetric with respect to indices 1 and 2, we find that

$$M_{12}=M_{21}$$

which is the reciprocity theorem.

(b) *Coefficient of coupling.* Just by definitions we have

$$M_{12}=\frac{\varphi_{12}}{i_1} \quad \text{and} \quad M_{21}=\frac{\varphi_{21}}{i_2}$$

and

$$L_1=\frac{\varphi_1}{i_1} \quad \text{and} \quad L_2=\frac{\varphi_2}{i_2}.$$

Let only a fraction (k_1) of φ_1 links with circuit 2 and a fraction k_2 of φ_2 with circuit 1. Then

$$\varphi_{12}=k_1\varphi_1 \quad \text{and} \quad \varphi_{21}=k_2\varphi_2$$

$$\therefore \quad \varphi_{12}\times\varphi_{21}=k_1k_2\varphi_1\varphi_2$$

or

$$(M_{21}\ i_1)\times(M_{21}\ i_2)=k_1k_2L_1i_1L_2i_2$$

or

$$M^2=k^2L_1L_2$$

where $k_1k_2=k^2$, a constant and $M_{12}=M_{21}$ by reciprocity theorem $=M$ (say)

or

$$k=\frac{M}{\sqrt{L_1L_2}}.$$

This constant k is called the coefficient of coupling. The greater the value of this constant, the greater is the mutual induction between the circuits. Its maximum value is 1 when coupling is so tight that the entire flux of 1 links 2 and the entire flux of 2 links 1.

4.3. Calculations of Self and Mutual inductance

(a) *Self-inductance of a circular coil.* Let us consider a circular coil of radius 'a' and carrying current i amp. The magnetic induction at the centre of the coil is

$$B=\frac{\mu_0\mu_r N i}{2a}$$

The magnetic flux linked with a turn of the coil is

$$\varphi=\Sigma\vec{B}.\Delta\vec{S}$$

$$=\Sigma B\Delta S\cos 0^\circ=\Sigma B\Delta S=B\Sigma\Delta S$$

(assuming B to be uniform throughout the area of the coil) or $$\varphi = BS = B \times \pi a^2, \qquad (\because \; S = \pi a^2)$$

$$= \frac{\mu_0 \mu_r N i}{2a} \times \pi a^2$$

Since self-inductance is the flux linked with unit current, therefore, self-inductance of 1 turn $= \frac{1}{2} \pi \mu_0 \mu_r N a$.

$\therefore$ L (self-inductance of the coil) $= N \times (\frac{1}{2} \pi \mu_0 \mu_r N a)$

or $$L = \tfrac{1}{2} \pi \mu_0 \mu_r N^2 a.$$

It is to be noted that this formula for self-inductance of a circular coil is not exact because the field at the centre has been assumed to be uniform throughout the area of the coil which is not true.

(b) *Self-inductance of a long solenoid.* Consider a solenoid of length l and having N turns. When the current through the solenoid is i, the magnetic induction at its centre is

$B = \mu_0 \mu_r n i$ where n = number of turns per unit length $= \frac{N}{l}$.

$$\therefore \qquad B = \frac{\mu_0 \mu_r N i}{l}.$$

The magnetic flux linked with one turn of the solenoid is

$$\varphi = \Sigma \vec{B} . \triangle \vec{S} = \Sigma B \triangle S \cos 0° = B \Sigma \triangle S = BS \qquad (\because \; \Sigma \triangle S = S)$$

$$\varphi = \left(\frac{\mu_0 \mu_r N i}{l} \right) S = \frac{\mu_0 \mu_r N S}{l} i$$

Since self-inductance is the flux linked with unit current, therefore, self-inductance of 1 turn of the solenoid $= \frac{\mu_0 \mu_r N S}{l}$.

$\therefore$ L (self-inductance of the solenoid) $= N \times \frac{\mu_0 \mu_r N S}{l} = \frac{\mu_0 \mu_r N^2 S}{l}$.

This formula is only approximate because the field at the centre has been assumed to be uniform over the entire length of the solenoid which is not true.

(c) *Self-inductance of long parallel wires.* Let P and Q be two long parallel wires such as the 'lead' and 'return' wires from a supply source to a load at some distance.

(c) *Mutual inductance of two coaxial solenoids.* Suppose P and S are two coaxial solenoids of turns N_1 and N_2 respectively. Let i be the instantaneous current through P. The magnetic field at the centre of P is

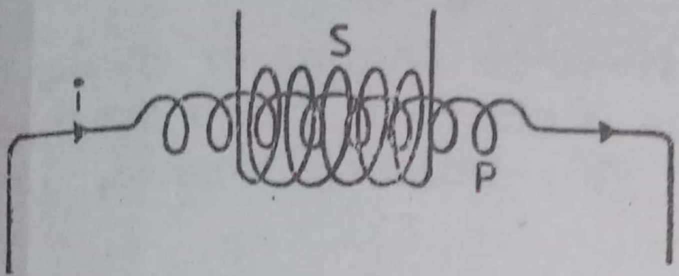


Fig. 4.6

$$B=\frac{\mu_0\mu_r N_1 i}{l}$$

where l=length of P.

The magnetic flux linked with one turn of S is

$$\varphi_s=\Sigma\vec{B}.\triangle\vec{S}=BS$$

$$=\frac{\mu_0\mu_r N_1 Si}{l}, (\because \quad \theta=0° \text{ and it is assumed}$$

that the field at the centre of the primary solenoid is uniform over the cross-section of the secondary solenoid).

∴ Mutual inductance per turn of $S = \frac{\mu_0 \mu_r N_1 S}{l}$

∴ Mutual inductance between P and $S = N_2 \times \frac{\mu_0 \mu_r N_1 S}{l}$

or
$$M = \frac{\mu_0 \mu_r N_1 N_2 S}{l} \text{ henry.}$$

6

MAXWELL'S EQUATIONS

The field considered upto chapter 4 are the static electric fields due to charges at rest. When charges are in motion, the electric and magnetic fields will be associated with this motion which will have space and time variations. The phenomenon is called electromagnetism and we study, under this head, the electro-magnetic wave motion and radiation. The set of equations, the so called Maxwell's equations, is merely the representation of laws of electromagnetism and describe the sources and field vectors in the broad fields of electrostatics, magnetostatics and electromagnetic induction. They enable us to examine the space and time variations of changing fields in the same way as the equations of electrostatics and magnetostatics permit an examination of stationary fields.

6·0. EQUATION OF CONTINUITY :

As we are now dealing with charges in motion, let us consider that charge density, ρ, is a function of time. The transport of charge constitutes the current *i.e.*

$$I=\frac{dq}{dt}=\frac{d}{dt}\int_V \rho \, dV, \qquad ...(1)$$

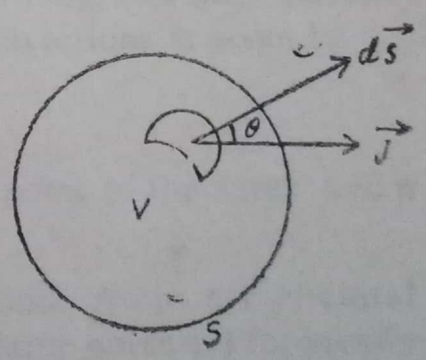


Fig. (1)

where we have considered that the current is extended in space of volume V closed by a surface S. The net amount of charge which crosses a unit area (normal to the direction of charge flow) of a surface in unit time is defined as the current density **J**. According to all experiments to date charge in a closed system is always *conserved*. Therefore if a net amount of current is flowing outward a closed surface, the charge contained within that volume should decrease, *i.e.*,

$$I = \frac{dq}{dt}, \qquad \ldots(2)$$

where I is the total current flowing through the surface **S**. If **J** is the current density, then by definition, total current I will be

$$I = \oint_S \mathbf{J} \cdot d\mathbf{S} \qquad \ldots(3)$$

From eqs. (2) and (3), we get

$$\oint_S \mathbf{J} . d\mathbf{S} = -\frac{dq}{dt}$$

$$= -\frac{d}{dt} \int_V \rho \, dV, \qquad \ldots(4)$$

using eq. (1).

Because it is ρ which is changing with time we can write

$$\frac{d}{dt} \int_V \rho \, dV = \int_V \frac{\partial \rho}{\partial t} \, dV,$$

so that eq. (4), becomes

$$\oint_S \mathbf{J} \cdot d\mathbf{S} = -\int_V \frac{\partial \rho}{\partial t} \, dV. \qquad \ldots(5)$$

From divergence theorem,

$$\oint_S \mathbf{J} \cdot d\mathbf{S} = \int_V (\text{div } \mathbf{J}) \, dV,$$

so that eq. (5), becomes

$$\int_V (\text{div } \mathbf{J}) \, dV = -\int_V \frac{\partial \rho}{\partial t} \, dV$$

or

$$\int_V \left(\text{div } \mathbf{J} + \frac{\partial \rho}{\partial t} \right) dV = 0. \qquad \ldots(6)$$

Since eq. (6) holds for any arbitrary volume, we can put integrand equal to zero, *i.e.*,

$$\text{div } \mathbf{J} + \frac{\partial \rho}{\partial t} = 0, \qquad \ldots(7)$$

and is referred to as the *equation of continuity*. It is the mathematical expression for the conservation of charge. It states that the '*total current flowing out of some volume must be equal to the rate of decrease of charge within the volume, assuming that charge can not be created or destroyed, i.e., no sources and sinks are present in that volume*'. In case of stationary currents, charge density at any point with in the region remains constant

$$\frac{\partial \rho}{\partial t}=0,$$

so that $\text{div}\,\mathbf{J}=0$

or $\nabla\,\mathbf{J}=0,$ ...(8)

which expresses the fact that there is no net outward flux of current density **J** *i.e.* lines of electric current are continuous.

6·1. DISPLACEMENT CURRENT :

Ampere's circuital law (eq. 5 Ex. 3 below art 5·8) is

$$\oint \mathbf{B}\cdot d\mathbf{l}=\mu_0\, I$$

or

$$\oint \mathbf{H}\cdot d\mathbf{l}=I$$

$$=\int_S \mathbf{J}\cdot d\mathbf{S} \text{ (art 6·0).}$$

Changing line integral into surface integral by Stoke's theorem,

$$\int_S \text{Curl}\,\mathbf{H}\cdot d\mathbf{S}=\int_S \mathbf{J}\cdot d\mathbf{S}$$

or $\text{Curl}\,\mathbf{H}=\mathbf{J}$

let us put it in equation of continuity which is

$$\text{div}\,\mathbf{J}=-\frac{\partial \rho}{\partial t}$$

$$\text{div (curl }\mathbf{H})=-\frac{\partial \rho}{\partial t}$$

$$0=-\frac{\partial \rho}{\partial t}.$$

That is, eq. (1) leads to steady state conditions in which charge density is not changing. Therefore for time dependent (changing) fields eq. (1) should be modified. Maxwell suggested that the definition of total current density is incomplete and advised to add something to **J**. Let it be **J′**. Then eq. (1) becomes

$$\text{Curl } \mathbf{H} = (\mathbf{J} + \mathbf{J}') \qquad \ldots(2)$$

In order to identify **J′**, we take divergence of eq. (2). That is

$$\text{div curl } \mathbf{H} = \text{div } (\mathbf{J} + \mathbf{J}')$$

$$0 = \text{div } \mathbf{J} + \text{div } \mathbf{J}'$$

or

$$\text{div } \mathbf{J}' = -\text{div } \mathbf{J}$$

$$= \frac{\partial \rho}{\partial t} \qquad \ldots(3)$$

We know that

$$\rho = \nabla . \mathbf{D}$$

so that eq. (3) becomes

$$\text{div } \mathbf{J}' = \frac{\partial}{\partial t} (\nabla . \mathbf{D})$$

$$= \nabla . \frac{\partial \mathbf{D}}{\partial t}$$

$$= \text{div} \left(\frac{\partial \mathbf{D}}{\partial t} \right)$$

or

$$\text{div} \left(\mathbf{J}' - \frac{\partial \mathbf{D}}{\partial t} \right) = 0 \qquad \ldots(4)$$

Since eq. (4) is true for any arbitrary volume, we can put

$$\mathbf{J}' = \frac{\partial \mathbf{D}}{\partial t} \qquad \ldots(5)$$

Therefore the modified form of the Ampere's law is

$$\text{curl } \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \qquad \ldots(6)$$

Since **J′** arises due to the variation of electric displacement **D** with time, it is termed as *displacement current* or *displacement current density*. According to Maxwell it is just as effective as **J**, the conduction current density, in producing magnetic field. The important inference that we draw from eq. (6) is that, since displacement current, **J′**, is related to the electric field vector **D** (as $\mathbf{D} = \epsilon \mathbf{E}$), it is not possible in case of time varying fields to deal separately with electric and magnetic fields but, instead, the two fields are interlinked giving rise to electromagnetic fields. Thus **J′** results into unification of electric and magnetic phenomenon. In a good conductor **J′** is negligible compared to **J** at frequency lower than light frequencies (10^{15} c/s) as is evident from the following example :

Example 1. *Show that for a conductor subject to electric field*

$$E=E_0 \cos \omega t$$

displacement current density is negligible compared to conduction current density at frequencies less than 10^{15} c/s.

We know that

$$\mathbf{D}=\epsilon\mathbf{E}=\epsilon E_0 \cos \omega t$$

so that

$$\mathbf{J}'=\frac{\partial \mathbf{D}}{\partial t}=-\epsilon\omega E_0 \sin \omega t$$

$$=\epsilon\omega E_0 \cos (\omega t+\pi/2)$$

According to Ohm's law, conduction current density is

$$\mathbf{J}=\sigma\mathbf{E}=\sigma E_0 \cos \omega t$$

so that

$$\frac{\mathbf{J}'}{\mathbf{J}}=\frac{\epsilon\omega}{\sigma}\frac{\cos (\omega t+\pi/2)}{\cos \omega t}$$

or

$$\left|\frac{\mathbf{J}'}{\mathbf{J}}\right|=\frac{\epsilon\omega}{\sigma}=\frac{2\pi\epsilon f}{\sigma}\approx 10^{-17} f$$

for a good conductor. Therefore for $f \lessdot 10^{15}$ c/s. $J' \ll J$ and negilgible. Or the ratio of conduction and displacement is invariably very high in the entire frequency range.

6·2. THE MAXWELL'S EQUATIONS (Differential Form)

We shall now state in differential form, the four equations of Maxwell :

(i) $\nabla.\mathbf{D}=\rho$ → results by the application of Gauss theorem to electrostatics. **D** is the electric displacement in coulombs/meter² and ρ is the free charge density in coulombs/meter³.

(ii) $\nabla.\mathbf{B}=0$ → results by the application of Gauss theorem to magnetic field. **B** is the magnetic induction in webers/meter².

(iii) $\nabla\times\mathbf{E}=-\frac{\partial \mathbf{B}}{\partial t}$ → results by Faraday's and Lenz's laws of electromagnetic induction. **E** is the electric intensity in volts/meter.

(iv) $\nabla\times\mathbf{H}=\mathbf{J}+\frac{\partial \mathbf{D}}{\partial t}$ → results by Maxwell's modification of Ampere's law in a circuital form for magnetic field accompanying an electric current. **H** is magnetic field intensity in amperes/meter. **J** is the current density in amperes/meter².

DERIVATION OF MAXWELL'S EQUATIONS :

(1) $\boxed{\text{div } \mathbf{D}=\rho}$.

Referring to Gauss's theorem, the integral $\int \mathbf{E}.d\mathbf{S}$ of the normal component of **E** over any closed surface is equal to the total charge enclosed within the surface. In a dielectric medium, as we have mentioned in the separate chapter of dielectrics, the total charge must include both the free and the polarisation charges or in other words volume density would be $\rho-(\text{div } \mathbf{P})$, *i.e.*,

$$\int \mathbf{E}.d\mathbf{S}=\int \text{div } \mathbf{E}\, dV=\int \frac{1}{\epsilon_0}(\rho-\text{div } \mathbf{P})\, dV$$

or
$$\int \text{div } (\epsilon_0\, \mathbf{E}+\mathbf{P})\, dV=\int \rho\, dV$$

The quantity $(\epsilon_0\, \mathbf{E}+\mathbf{P})$ is denoted by a quantity **D**, called the *electric displacement*, *i.e.*,

$$\int \text{div } \mathbf{D}\, dV=\int \rho\, dV$$

or
$$\int \left(\text{div } \mathbf{D}-\rho\right) dV=0.$$

Since the equation is true for all volumes, the integrand in this equation must vanish, *i.e.*,

$$\text{div } \mathbf{D}=\rho.$$

When the medium is isotropic the three vectors **D, E, P** are in the same direction and for small field, **D** is proportional to **E**. *i.e.*,

$$\mathbf{D}=\epsilon\mathbf{E},$$

where ϵ is called dielectric constant of the medium.

(2) $\boxed{\text{div } \mathbf{B}=0}$

Since the magnetic lines of force are either closed or go off to infinity, the number of magnetic lines of force entering any arbitrary close surface is exactly the same as leaving it. It means the flux of magnetic induction **B** across any closed surface is always zero, *i.e.*,

$$\int \mathbf{B}.d\mathbf{S}=0.$$

Transforming the surface integral into volume integral, we have

$$\int \text{div } \mathbf{B}\, dV=0.$$

The integrand should vanish for the surface boundary as the volume is arbitrary, *i.e.*,

$$\text{div } \mathbf{B}=0.$$

(3) $\boxed{\text{Curl } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}}$

By Faraday law, we know that e.m.f. induced in a closed loop is given by

$$e = -\frac{\partial \phi}{\partial t} = -\int \frac{\partial \mathbf{B}}{\partial t} . d\mathbf{S},$$

since the flux $\phi = \int_S \mathbf{B} . d\mathbf{S}$ where S is any surface having the loop as boundary.

e.m.f., e, can also be found by calculating the work done in carrying a unit charge completely around the loop. Thus

$$e = \oint \mathbf{E} . \mathbf{dl},$$

$\mathbf{E}$ being the intensity of the field associated with induced e.m.f.

Therefore, equating above two equations, we get

$$\oint \mathbf{E} . \mathbf{dl} = -\int_S \frac{\partial \mathbf{B}}{\partial t} . d\mathbf{S}.$$

Applying Stoke's theorem, the line integral can be transformed into surface integral, *i.e.*,

$$\int_S (\nabla \times \mathbf{E}) . d\mathbf{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t} . d\mathbf{S}.$$

This equation must be true for any surface whether small or large in the field. Therefore the two vectors in the integrands must be equal at every point, *i.e.*,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

or

$$\text{Curl } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

(4) $\boxed{\text{Curl } \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}}$.

Ampere's law in the circuital form yields this equation. According to this law, the work done in carrying a unit magnetic pole once around a closed arbitrary path linked with the current is expressed by

$$\oint \mathbf{H} . \mathbf{dl} = I.$$

or $$=\int \mathbf{J}.d\mathbf{S},$$

where the integral on the right is taken over the surface through which the charge flow corresponding to the current I takes place.

Now changing the line integral into surface integral by Stoke's theorem,

$$\int_S \text{Curl } \mathbf{H}.d\mathbf{S}=\int \mathbf{J}.d\mathbf{S}.$$

$$\text{Curl } \mathbf{H}=\mathbf{J}.$$

The above relation, derived on the basis of Ampere's law, stands only for steady closed current but for the changing electric fields, the current density should be modified. The difficulty with above equation is that if we take divergence of above equation then

$$\text{div (curl } \mathbf{H})=0,$$

or div $\mathbf{J}=0$ which conflicts with the equation of continuity div $\mathbf{J}=-\partial\rho/\partial t$. Therefore Maxwell realised that the definition of total current density is incomplete and suggested to add something to $\mathbf{J}$, *i.e.*,

$$\text{Curl } \mathbf{H}=(\mathbf{J}+\mathbf{J}').$$

Now taking divergence of above equation, we get

$$\text{div (curl } \mathbf{H})=(\text{div } \mathbf{J}+\text{div } \mathbf{J}'),$$

or $$0=\text{div } \mathbf{J}+\text{div } \mathbf{J}'$$

or $$\text{div } \mathbf{J}'=-\text{div } \mathbf{J}=+\frac{\partial\rho}{\partial t}.$$

We know that $\rho=\nabla.\mathbf{D}$.

Putting this value in the expression for div $\mathbf{J}'$, we get

$$\text{div } \mathbf{J}'=\frac{\partial}{\partial t}(\nabla.\mathbf{D})$$

$$\nabla.\mathbf{J}'=\nabla.\left(\frac{\partial \mathbf{D}}{\partial t}\right).$$

Hence $$\mathbf{J}'=\frac{\partial \mathbf{D}}{\partial t}.$$

Therefore the Maxwell's fourth relation can be written as

$$\text{Curl } \mathbf{H}=\mathbf{J}+\frac{\partial \mathbf{D}}{\partial t}.$$

It is quite obvious from the relation for $\mathbf{J}'$ that the term $\partial\mathbf{D}/\partial t$ on right hand side arises when the electric displacement $\mathbf{D}$ is changing with time and is, therefore, termed as *Displacement current density*. According to Maxwell, it is just as effective as $\mathbf{J}$ in producing magnetic field.

WORD STATEMENT OF THE FIELD EQUATIONS :

(Maxwell's equations in integral form) :

The significance of the field equations is readily obtained from their mathematical statement in integral form.

→ (1) **For equation**

$$\nabla . \mathbf{D} = \rho.$$

Integrating this equation over a volume V, we arrive at

$$\int_V \nabla . \mathbf{D} dV = \int_V \rho \, dV.$$

But from Gauss theorem, we get

$$\int_S \mathbf{D} . d\mathbf{S} = \int_V \rho \, dV$$
$$= q,$$

where q is the net charge contained in volume V. S is the surface bounding volume V.

Therefore this Maxwell's equation signifies that :

The total electric displacement through the surface enclosing a volume is equal to the total charge within the volume.

→ (2) **For equation** $\nabla . \mathbf{B} = 0.$

Exactly in a manner adopted above, we can arrive at

$$\int_S \mathbf{B} . d\mathbf{S} = 0.$$

which signifies that :

The total outward flux of magnetic induction **B** *through any closed surface S is equal to zero.*

→ (3) **For equation** $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$

Integrating this equation over a surface S, bounded by a curve we arrive at

$$\int_S (\nabla \times \mathbf{E}) . d\mathbf{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t} . d\mathbf{S}.$$

Converting the surface integral of left hand side into line integral by Stoke's theorem, we get

$$\oint \mathbf{E} . d\mathbf{l} = -\frac{\partial}{\partial t} \int_S \mathbf{B} . d\mathbf{S}$$

which signifies that :

The electromotive force around a closed path is equal to the time derivative of the magnetic displacement through any surface bounded by the path.

→ (4) **For equation** $\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$.

which can written in integral form as

$$\oint \mathbf{H} . d\mathbf{l} = \int_s \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) . d\mathbf{S},$$

on proceeding exactly in the similar fashion as adopted in (3). This form signifies that :

The magnetomotive force around a closed path is equal to the conduction current plus the time derivative of the electric displacement through any surface bounded by the path.

The correspondence of **B** and **H** with **E** and **D** through the Maxwell's curl equations (3) and (4) implies that time varying electric and magnetic fields in empty space are interdependent *i.e.*, a changing electric fieid being able to generate a magnetic field and vice versa. From this we draw the inference that a *time-changing electromagnetic field would propagate energy through empty space with the velocity of light* (see next chapter) *and further, that light is electromagnetic in nature.*

Example 1. *Using Maxwell's relation*

$$\text{curl } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \text{ and curl } \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

show that

$$\text{div } \mathbf{B} = 0 \text{ and div } \mathbf{D} = \rho$$

Taking divergence of both the given Maxwell's equations we get

$$\text{div curl } \mathbf{E} = -\text{div} \left(\frac{\partial \mathbf{B}}{\partial t} \right)$$

or $$0 = -\text{div} \left(\frac{\partial \mathbf{B}}{\partial t} \right)$$

or $$0 = -\frac{\partial}{\partial t} (\text{div} \mathbf{B})$$

or $$0 = -\frac{\partial}{\partial t} (\nabla . \mathbf{B}) \quad ...(1)$$

$$\text{div curl } \mathbf{H} = \text{div} \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right)$$

$$0 = \text{div } \mathbf{J} + \frac{\partial}{\partial t} (\nabla . \mathbf{D})$$

$$\text{div } \mathbf{J} = -\frac{\partial}{\partial t} (\nabla . \mathbf{D})$$

$$-\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial t} (\nabla . \mathbf{D}) \quad ...(2)$$

using eq. of continuity.

If for each point of space div **B** and div **D** become zero at any time either in past or future, then eqs. (1) and (2) will give

$$\text{div } \mathbf{B} = 0 \text{ and div } \mathbf{D} = \rho.$$

Example 2. *Using Maxwell's relation*

$$\text{div } \mathbf{D} = \rho \text{ and curl } \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

prove

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(i) $$\mathbf{F}=\frac{1}{4\pi\epsilon_0}\frac{q\,q_0}{r^3}\,\mathbf{r}, \quad \textit{Coulomb's law}$$

and (ii) $$\text{div } \mathbf{J}+\frac{\partial \rho}{\partial t}=0, \quad \textit{equation of continuity.}$$

(i) To prove Coulomb's law :

Integrating over a sphere of radius r the relation

$$\text{div } \mathbf{D}=\rho,$$

we get

$$\int_V \text{div } \mathbf{D}\, dV=\int_V \rho\, dV$$

$$=q, \text{ the total charge.}$$

Changing volume integral into surface integral,

$$\int_S \mathbf{D}.d\mathbf{S}=q$$

$$\mathbf{D}\,(4\pi r^2)=q$$

$$\epsilon_0\,\mathbf{E}\,(4\pi r^2)=q$$

$$\mathbf{E}=\frac{q}{4\pi\epsilon_0\, r^2}$$

Force on a test charge q_0 will be

$$\mathbf{F}=q_0\,\mathbf{E}$$

$$=\frac{1}{4\pi\epsilon_0}\frac{q\,q_0}{r^3}\,\mathbf{r}.$$

(ii) To prove equation of continuity :

Taking divergence of eq.

$$\text{curl } \mathbf{H}=\mathbf{J}+\frac{\partial \mathbf{D}}{\partial t},$$

we get

$$\text{div curl } \mathbf{H}=\text{div}\left(\mathbf{J}+\frac{\partial \mathbf{D}}{\partial t}\right)$$

$$0=\text{div } \mathbf{J}+\text{div}\left(\frac{\partial \mathbf{D}}{\partial t}\right)$$

or $$\text{div } \mathbf{J}+\frac{\partial}{\partial t}(\text{div } \mathbf{D})=0$$

or $$\text{div } \mathbf{J}+\frac{\partial \rho}{\partial t}=0$$

6·3. (A) MAXWELL'S EQUATIONS IN FREE SPACE :

In the special case of free space, where the current density $\mathbf{J}$ and volume charge density ρ are zero, Maxwell's equations reduce to

$$\left.\begin{aligned} \nabla.\mathbf{D}&=0\\ \nabla.\mathbf{B}&=0\\ \nabla\times\mathbf{E}&=-\frac{\partial\mathbf{B}}{\partial t}\\ \nabla\times\mathbf{H}&=\frac{\partial\mathbf{D}}{\partial t}\end{aligned}\right\} \quad ...(1)$$

(B) Maxwell's Equations in Linear Isotropic Media :

In linear isotropic media,

$$\mathbf{D}=\epsilon\mathbf{E}$$

and

$$\mathbf{H}=\frac{\mathbf{B}}{\mu}$$

where ϵ is dielectric constant,

μ permeability of the medium.

The Maxwell's equations become

$$\left.\begin{aligned} \nabla.\mathbf{E}&=\frac{\rho}{\epsilon}\\ \nabla.\mathbf{H}&=0\\ \nabla\times\mathbf{E}+\mu\frac{\partial\mathbf{H}}{\partial t}&=0\\ \nabla\times\mathbf{H}-\epsilon\frac{\partial\mathbf{E}}{\partial t}&=\mathbf{J}\end{aligned}\right\} \quad ...(2)$$

(C) Maxwell's Equations for Harmonically Varying Fields :

If we assume that the fields vary harmonically with time, Maxwell's equations can be expressed in another special form.

$$\mathbf{D}=\mathbf{D}_0\, e^{j\omega t}$$

then

$$\frac{\partial\mathbf{D}}{\partial t}=j\omega\mathbf{D}_0\, e^{j\omega t}$$

$$=j\omega\mathbf{D}.$$

Similarly, we can write

$$\frac{\partial\mathbf{B}}{\partial t}=j\omega\mathbf{B}$$

The Maxwell's equations becomes

$$\left.\begin{aligned} \nabla.\mathbf{D}&=\rho\\ \nabla.\mathbf{B}&=0\\ \nabla\times\mathbf{E}+j\omega\mathbf{B}&=0\\ \nabla\times\mathbf{H}-j\omega\mathbf{D}&=\mathbf{J}\end{aligned}\right\} \quad ...(3)$$

6·4. ENERGY IN ELECTROMAGNETIC FIELDS : POYNTING VECTOR (Poynting Theorem) :

Energy may be transported through space by means of electro-